Efficient and Sustainable Risk Sharing with Adverse Selection[*]

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Abstract

We analyze a model with two risk averse agents who engage in risk sharing over an infinite time horizon in the presence of asymmetric information: one agent's stochastic income realizations are his private information. Both agents can opt out of the risk sharing agreement at any time. We prove existence and uniqueness of efficient and incentive compatible sharing rules. We find inefficient risk sharing across states at all times, and show history dependence of efficient sharing rules: consumption strictly increases over time for an agent who is repeatedly hit with a favorable income shock. Under some restrictions on agents' utility functions, we show that after a better history, an agent's consumption is strictly larger. Agents' consumption is therefore positively correlated with past income. We argue that these results can explain empirical findings of excess sensitivity to income changes and history dependence of agents' consumption. We also prove the existence of a unique stationary wealth distribution in which neither agent becomes impoverished. We provide an algorithm for numerical computation and compute examples for agents with CRRA utility and various values of risk aversion and impatience.

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1 Introduction

Risk averse agents, facing idiosyncratic risk, benefit from insurance to reduce the consumption uncertainty at a given time period and the variability of consumption over time. Given the level of aggregate uncertainty, the evidence from household consumption data suggests that the extent of risk sharing is below the optimal level that is consistent with the complete markets hypothesis. Consumption displays excess sensitivity to income shocks, and agents' optimal consumption plans depend on the histories of their individual incomes.

The possibility that agents' abilities to commit to a transfer scheme are limited has for some time been considered a likely explanation for these phenomena. We argue that no commitment is unlikely to address the empirical evidence by itself. We present a model without commitment and asymmetric information, and show that the resulting economy displays many of the features found in the data.

The informational asymmetry we consider is unobservability of an agent's random income. Our model therefore applies to various risk sharing agreements such as risky debt contracts with entrepreneurs, labor hoarding by employers (where the employee's productivity is unobservable), or informal loans between members of a social group. We also assume that agents are unable to commit to a contract. The no commitment assumption is motivated by bankruptcy laws and lack of enforcement mechanisms in informal arrangements.

To illustrate our approach, consider the following infinitely repeated exchange economy: there are two agents, 1 and 2, and two states, H and L. State realizations are independent over time, and state H occurs with probability $\frac{1}{2}$. Both agents have square-root utility $u(c) = \sqrt{c}$ and a common discount factor δ. Each period, agent 1 receives 1 unit of some consumption good in state H, and 0 units in state L. Agent 2 receives a single unit of the consumption good in both states. Though the consumption good is not storable, agents can write contracts among themselves to smooth their consumption. At any history of state realizations $h_t = (s_1, ..., s_t)$, a contract c specifies agent 1's consumption $c(h_t)$. Agent 2's consumption is then $\omega^{s_t} - c(h_t)$, where ω^s is the economy's aggregate income in state s :

$$\omega^s = \begin{cases} 2, & \text{if } s = H \\ 1, & \text{if } s = L. \end{cases}$$

$U^i(c, h_{t-1})$ refers to agent i's expected utility from the contract c at history h_{t-1}, before the state is known; $U^i_s(c, h_{t-1})$ is his expected utility at the same date after the state is revealed to be s. The contract in which no transfers are ever made is denoted c^*. In this contract, agents always receive their autarky consumption: $c^*(h_{t-1}, H) = 1$, $c^*(h_{t-1}, L) = 0$ at all h_t, where (h_{t-1}, s) refers to the $t-$period history in which s follows h_{t-1}.

In our simple economy, straightforward calculations show that for any efficient contract under complete contracting, there exists a tuple (c_H, c_L) with $c_H = 2c_L$, such that $c(h_{t-1}, H) = c_H$, $c(h_{t-1}, L) = c_L$ for all h_{t-1}. Any such contract is feasible that yields both agents at least their autarky utility ex ante.

Kocherlakota (1996) and Ligon et al. (2002) analyze optimal contracts in an infinitely repeated exchange economy without commitment to explain observed consumption inefficiencies. Interpreting the no commitment assumption as a borrowing constraint, Kehoe and Levine (1993, 2001) and Alvarez and Jermann (2000, 2001) analyze asset markets in similar economies. In these models, agents are unable to commit to a contract[1]. If an agent has to give up so much of his endowment in some state s_t that he prefers to walk away from the contract and revert to consuming his income stream, the contract is not feasible. More formally, any feasible contract without commitment must satisfy the participation constraints

$$U^i_s(c, h_{t-1}) \geq U^i_s(c^*, h_{t-1}), \text{ for } s \in \{H, L\}, i \in \{1, 2\}, \tag{1}$$

at all h_{t-1}. It can be shown that an efficient contract in an economy without commitment is a tuple $(c_0,\ s,\ c_H,\ c_L)$, $s \in \{H, L\}$, such that $c(h_{t-1}, s) = c_0$ for any $h_{t-1} = (s, s, ..., s)$, and $c(h_{t-1}, H) = c_H$, $c(h_{t-1}, L) = c_L$ for all other h_{t-1}. Therefore, after the first state transition, the contract is fully determined by $(c_H,\ c_L)$. If δ is large enough, a first-best contract $c_H = 2c_L$ is sustainable. For smaller δ, agents' consumption in their high income state has to be increased, in order for them to be willing to share their high income. The unique first-best contract then has $c_H > 2c_L$. Agents' participation constraints (1) bind in states where they are called upon to make a transfer:

$$U^1_H(c, h_{t-1}) = U^1_H(c^*, h_{t-1}) \text{ and } U^2_L(c, h_{t-1}) = U^2_L(c^*, h_{t-1}).$$

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[1] Similar models with complete information and no commitment are analyzed in Thomas and Worrall (1988), Dixit, Grossman and Gul (2000), and Kletzer and Wright (2000).

Under complete contracting, agents' consumption satisfies (i) $c(h_{t-1},\, s_t) = c(h_{t-1},\, s_t, s_{t+1})$ for $s_t = s_{t+1}$ and (ii) $c(h_{t-1},\, H) = 2c(h_{t-1},\, H,\, L)$, $2c(h_{t-1},\, L) = c(h_{t-1},\, L,\, H)$. If consumption satisfies both (i) and (ii), we refer to this as *efficient consumption smoothing*. Intertemporal relations like (i) and (ii) are easily observable, and are taken as starting points for most tests of the complete markets hypothesis. In the data, consumption usually reacts more strongly to income changes than predicted by a complete markets model (for example Cochrane (1996), Nelson (1994), or Townsend (1994), Altonji et al. (1992)). These effects are not always large in magnitude, but are significant over time, as consumption is consistently found to vary too much with current income. This phenomenon is the complete markets analogue of *excess sensitivity* to transitory income shocks as found by Hall and Mishkin (1982). Hall and Mishkin test the permanent income hypothesis, in which the agent's option is to self-insure, and find that transitory income shocks are not fully insured. These findings are similar enough in spirit to the evidence from tests of the complete markets hypothesis that we will in the following borrow their term to describe the overreaction of agents' consumption to transitory income shocks. In our example, agent 1's consumption exhibits *intertemporal excess sensitivity* to an income change whenever $2c(h_{t-1}, L) < c(h_{t-1}, L, H)$ or $c(h_{t-1}, H) > 2c(h_{t-1}, H, L)$. Due to the aggregate resource constraint, excess sensitivity of agent 2's consumption is implicit in the above definition. Intertemporal excess sensitivity is an indicator of suboptimal risk sharing across states, $c(h_{t-1}, H) > 2c(h_{t-1}, L)$, which is in general not observable. We refer to $c(h_{t-1}, H) > 2c(h_{t-1}, L)$ as *cross-sectional excess sensitivity*.

As can be seen from our characterizations of optimal contracts, an economy without commitment will only exhibit excess sensitivity if agents are relatively impatient. Recall that in our example, a sustainable contract is a tuple $(c_H,\, c_L)$ after the first state transition. If δ is small enough, $c_H > 2c_L$: consumption paths are constant as long as the state remains constant; if the state changes, consumption changes by more than warranted by efficient consumption smoothing.

Another important implication of complete contracting is that consumption levels are history-independent, that is, depend only on payoff-relevant aspects of income histories. However, the empirical evidence suggests that an agents' income history does influence his current consumption level (Altonji et al. (1992), Townsend (1994), or Fafchamps and

Lund (2003)). Note that history dependence is not a necessary consequence of excess sensitivity. In our example, an efficient contract in the economy without commitment is a tuple (c_H, c_L), such that $c(h_{t-1}, H) = c_H$, $c(h_{t-1}, L) = c_L$ for all h_{t-1}. If δ is low enough, efficient risk sharing is not sustainable, and $c_H > 2c_L$. Consumption displays excess sensitivity to a change in income, but it is completely determined by current income for any history. Therefore, there is never any history dependence in this economy. In the data, history dependence and excess sensitivity coexist.

In our model, history dependence can be captured in two distinct ways. At any given history h_t, an agent's optimization problem should be identical to the one he faces in period $t + 1$, if his income stays constant. If an additional identical income realization affects his consumption, we refer to it as *intertemporal history dependence*. That is, for histories h_t, h_{t+1} such that $h_{t+1} = (h_t, s_{t+1})$ and $s_{t+1} = s_t = H(L)$, consumption exhibits intertemporal history dependence if $c(h_t) < (>)\ c(h_{t+1})$. Alternatively, for a given history, we can ask whether changing a past income realization would have an effect on agents' consumption. We refer to this as *cross-sectional history dependence*. Consumption is defined to be *cross-sectionally history dependent* if $c(h_t) > c(h'_t)$, for two histories h_t, h'_t, that only differ at $t - k$, such that $s_{t-k} = H$, $s'_{t-k} = L$, for $k \geq 1$.

In the no commitment economy with two states, history dependence is always absent. In general, optimal contracts without commitment feature efficient consumption smoothing whenever sustainable (so that consumption is always constant if the state does not change). With a more general income process, this can lead to cross-sectional history dependence, under some special circumstances. For two histories h_t, h'_t that differ only at $t - k$, consumption can only be history dependent if either s_{t-k} or s'_{t-k} is the most extreme income level in the sequence (s_{t-k}, ..., s_t) or (s'_{t-k}, ..., s'_t), respectively. If the state changes, a first-best contract will change agents' consumption to the efficient counterpart of their previous consumption levels (in our example, $2c_L$ or $\frac{1}{2}c_H$). The new consumption level is then completely determined by current aggregate income and previous consumption, which is determined by previous income. The no commitment model features such consumption paths whenever they are sustainable, i.e. if the first-best consumption of the agent making the transfer is larger than his minimal sustainable consumption in that state. In this case, previous income has persistent effects on current consumption. However, on such a history path, consumption never displays excess sensitivity, as consumption smoothing is efficient. Intuitively, the model without commitment can deliver either excess sensitivity or cross-sectional history dependence, but not both. Consider consumption at t and $t+1$. If an agent's participation constraint binds at $t+1$, his income history prior to $t+1$ becomes irrelevant. On the other hand, his consumption can only display excess sensitivity at $t+1$ if his participation constraint binds at at $t+1$.

In this paper, we assume that agent 1's income is not observed by agent 2. Agents cannot commit to a contract. An efficient and sustainable contract must now ensure that it is always in agent 1's interest to announce his income correctly . If $s_t = H$, he must prefer to announce this, instead of consuming his high income as well as the transfer from agent 2. Therefore, in addition to agents' participation constraints (1), c must satisfy the revelation constraint

$$\sqrt{c(h_{t-1}, H)} + \delta U^1(c, h_t) \geq \sqrt{c(h_{t-1}, L) + 1} + \delta U^1(c, h'_t)$$

at all histories h_{t-1}, with $h_t = (h_{t-1}, H)$, $h'_t = (h_{t-1}, L)$.

In Proposition 2, we show that the presence of the revelation constraint implies $c(h_{t-1}, H) > 2c(h_{t-1}, L)$ at all histories: agents' consumption always exhibits cross-sectional excess sensitivity. The informed agent has to be rewarded for revealing the high state correctly by allowing him more consumption in his high income state than implied by efficient risk sharing. In any model where agents can optimize, if risk sharing is optimal across states, there is usually no impediment for efficient consumption smoothing over time. Efficient consumption smoothing, if the state at t is H, requires $c(h_{t-1}, H) = c(h_{t-1}, H, H)$ (no intertemporal history dependence) and $c(h_{t-1}, H) = 2c(h_{t-1}, H, L)$ (no intertemporal excess sensitivity). Clearly this is only possible if $c(h_{t-1}, H, H) = 2c(h_{t-1}, H, L)$, that is, there is no cross-sectional excess sensitivity. Though risk sharing across states is in general not observable, cross-sectional excess sensitivity is a necessary condition for any model that intends to explain observed inefficiencies in the allocation of risk.

Proposition 3 then shows that in an efficient contract, agent 1's consumption will always display intertemporal excess sensitivity to an unfavorable income shock:

$$c(h_{t-1}, H) > 2c(h_{t-1}, H, L) \text{ for all } h_{t-1}.$$

Additionally, an agent will experience strictly positive consumption growth if he is hit by successive favorable income shocks (intertemporal history dependence):

$$c(h_{t-1}, H) < c(h_{t-1}, H, H) \text{ and } c(h_{t-1}, L) > c(h_{t-1}, L, L),$$

if participation constraints do not bind at h_{t-1}. Consumption growth is limited above and below by the presence of agents' participation constraints. As consumption shows cross-sectional excess sensitivity tomorrow, the smoothest consumption profile will have today's consumption $c(h_{t-1}, H)$ lying strictly between tomorrow's values, relative to aggregate income, or $c(h_{t-1}, H, H) > c(h_{t-1}, H) > 2c(h_{t-1}, H, L)$. Agent 1 is rewarded for sharing his high income in t by increasing his consumption at t as well as his expected consumption in $t+1$. To induce him to share the additional high income in $t+1$, he must then be rewarded by even higher consumption. A similar argument applies if today's state is L. However, to achieve efficient incentives, agent 1's is usually forced to consume too much in the low state, while his future consumption after state L is decreased. This diminishes his incentive to cheat by making additional consumption today relatively less attractive, as his utility function is concave. If today's state is L, the smoothest allocation is then not always efficient, so that we have $c(h_{t-1}, L) > c(h_{t-1}, L, L)$.

In comparing the no commitment economy with binding participation constraints (small δ) with the economy with asymmetric information, one now observes a very different pattern of consumption reactions to income changes. Let (c_H, c_L) again denote the efficient contract under no commitment and complete information. To illustrate this point, we isolate an income change by looking at the following history: suppose in period t, the state has been L for a number of periods, and will be H for the next $k \geq 2$ periods. In the complete information economy, consumption immediately adjusts to this income change by jumping to c_H as soon as the state changes, and stays constant afterwards. Under asymmetric information, consumption might not react immediately (relative to aggregate income), if the history is such $c(h_{t-1}, L, H) = 2c(h_{t-1}, L)$. However, it will then start to increase gradually towards the new income level. For the reverse income change from H to L, agent 1's consumption reacts immediately, but shows the same gradual decrease. Consumption therefore reacts sometimes slowly, but always smoothly, to changes in individual income. The slow comovement of consumption with income is therefore in stark contrast to the immediate and complete adjustment implied by excess sensitivity under complete information, and seems closer to the empirical evidence.

Proposition 4 shows that, under certain restrictions on agents' utility functions, a favorable income change will permanently increase an agent's consumption level, unless the consumption process has become trapped at an agent's participation constraint. We therefore only compare histories that do not become trapped at an agent's participation constraint. We show that for two such histories h_t, h'_t, that only differ at $t-k$, where h_t is more fortunate for agent 1 ($s_{t-k} = H$, $s'_{t-k} = L$), his consumption is strictly higher at h_t: $c(h_t) > c(h'_t)$ (cross-sectional history dependence). For incentive reasons, agents' consumption must increase in response to favorable income shocks; but as agents are risk averse, this increase is smoothed over current and future consumption, so that income has persistent effects on agents' consumption.

In the model with complete information and no commitment, consumption is only history dependent if agents' participation constraints do not bind; in this case, agents are able to achieve efficient consumption smoothing. History dependence comes at the expense of excess sensitivity. Under asymmetric information, history dependence and excess sensitivity coexist, as is found in the data.

Proposition 5 characterizes the consumption process if agent 2's participation constraint binds. Let h_t, h'_t be two histories such that h_t is more fortunate for agent 1 than h'_t. We show that if agent 2's participation constraint binds at h_t as well as at h'_t and $s_t = L$, agent 1's consumption is strictly lower at h_t than at $h'_t : c(h_t) < c(h'_t)$. His future consumption, however, is higher at the more fortunate history h_t. Recall that in the low state, agent 1 is usually given too much current consumption at the expense of future consumption, to make the immediate consumption gain from cheating less attractive. His consumption profile in the low state is therefore distorted downwards, as long as this distortion is sustainable. When agent 2's participation constraint starts to bind, the distortion is no longer sustainable and has to be undone, if agent 1's expected utility is to be increased even further. The more fortunate history h_t thus leads to strictly smaller low state consumption than h'_t, but better future prospects, if agent 2's participation constraint binds at both histories. In the model with complete information and no commitment, the economy displays amnesia once an agent's participation constraint binds: if two histories both cause the same agent's participation constraint to bind, previous income histories cease to matter. Under asymmetric information, the economy can keep track of an agent's income histories, even at a participation constraint.

In all models considered here, agents' utilities follow a Markov process. To characterize the stationary distribution (if any) of these processes has long been a focus of interest in the risk sharing literature. We refer to the stationary distribution of the utility process as a wealth distribution. Atkeson and Lucas (1992) and Thomas and Worrall (1990) analyze risk sharing in economies with adverse selection and full commitment. They show that players become impoverished in the long run, as their utilities converge to the lowest possible value. This is mainly due to the fact that they assume agents' utilities to be unbounded below, so that the desired incentives can always be achieved by decreasing low-state consumption. The data do not seem to indicate such a unidirectional progression of inequality. In fact, Townsend (1994) finds that the composition of consumption distributions can change noticeably within a generation. In bounded economies, the wealth distribution is in general non-degenerate, as is found in Kocherlakota (1996)[2]. However, Ligon et al. (2002) find that the model with no commitment and complete information, when calibrated to yield excess sensitivity close to the data, predicts too high a degree of wealth inequality.

For both agents in our model, there is a wealth level at which one of them is impoverished, in the sense of only achieving his autarky utility. In Proposition 6, we show the existence of a unique stationary wealth distribution which puts strictly positive weight on both of these wealth levels. The wealth distribution is therefore non-degenerate, and shows that agents' fortunes may change, as is found in the data. Moreover, we show that the stationary distribution puts strictly positive weight on utility values where neither agent is impoverished. The consumption distribution is therefore more equitable under asymmetric information, which is desirable in light of the empirical evidence.

Phelan and Townsend (1991) provide a general method for computing numerical results in economies with asymmetric information, where the constraint set is not convex. Their method involves the use of lotteries as a convexification device. We show in Lemma 1 that we can derive the necessary convexity properties of the constraint set by assuming that agents' absolute risk aversion is nonincreasing. In Proposition 7, we utilize this result to define an algorithm without lotteries and prove its convergence.

We compute optimal contracts for agents with CRRA utility and various values of risk aversion and impatience. We calculate consumption statistics to show the extent

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[2] A related but weaker result is obtained in Wang (1995). He analyzes a bounded exchange economy with symmetric incomplete information and full commitment on both sides. He shows that agents' utilities do not converge.

of history dependence occurring in agents' consumption processes. Covariances of consumption with lagged income are positive and decrease in magnitude with the length of the lag. We compute stationary wealth distributions and show that for almost all parameter values, agents' borrowing constraints bind less than 1% percent of the time.

2 The Model

We analyze an economy with two agents who engage in risk sharing over an infinite time horizon. They each face an income stream of a non-storable consumption good. One of the agents has a fixed income every period, while the other's is uncertain. There are two states of the world, H and L. Agent 2 has a state-independent income of 1 every period. In state $s = L$, agent 1 has an endowment of 0, while in state H, his endowment is e. We assume $0 < e \leq 1$.[3] The economy's state-dependent aggregate income is therefore

$$\omega^s = \begin{cases} 1, \text{ if } s = L \\ 1 + e, \text{ if } s = H \end{cases}.$$

We assume that states are independent and identically distributed. In any period t and for any sequence of prior realizations $(s_1, ..., s_{t-1})$, the probability that $s_t = H$ is $\pi \in (0, 1)$. Only agent 1 observes his income (i.e. the state) each period.

Agents' instantaneous utility functions u_i are increasing and strictly concave; we also require them to exhibit nonincreasing absolute risk aversion, that is, for $x, y, z \in \mathbb{R}, w > 0$ and $\lambda \in (0, 1)$,

$$\begin{aligned} \lambda u_i(x) + (1-\lambda)u_i(y) &\geq u_i(z) \text{ implies} \\ \lambda u_i(x+w) + (1-\lambda)u_i(y+w) &\geq u_i(z+w), \end{aligned}$$

$i = 1, 2$. The u_i's are bounded on the relevant interval[4]: $\lim_{c\to 0} u_i(c) > -\infty$, $i = 1, 2$. Agents discount the future with discount factor $\delta \in (0, 1)$ and maximize the sum of discounted expected payoffs.

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[3] If agent 1's information has too large an effect on the economy (e is disproportionally large), corner solutions are possible. The weaker assumption $\frac{u_1'(e)}{u_1'(1)} \geq \frac{u_2'(1)}{u_2'(0)}$ is actually sufficient to avoid this (see Section 6).

[4] Alternatively, we could assume agent 1's low-state income to be strictly greater than zero, as is done in much of the literature. All results are unchanged in this case (which we assume in Section 6).

If agents enter into a risk sharing agreement, they give some of their income to their opponent in states where they have a relatively high income, with the understanding that the opponent will return the favor when his luck changes. Such an agreement is desirable as aggregate endowment is uncertain, and saving not possible.

To formally define the notion of a risk sharing agreement, take a sequence $(s_1, ..., s_t)$ of state realizations. We shall refer to such finite sequences as $t-$period histories. For $t \geq 1, \mathcal{H}_t := \{H, L\}^t$ denotes the set of all $t-$period histories, and $\mathcal{H} = \bigcup_{t=1}^{\infty} \mathcal{H}_t \cup \{h_0\}$ the set of all histories, including the null history. Given our independence assumption, the probability of any history h_t can be computed as

$$p(h_t) = \pi^{\#h_t}(1-\pi)^{t-\#h_t},$$

where $\#h_t$ denotes the number of times the state H occurred in the history h_t.

We define a sharing rule $c : \mathcal{H} \to \mathbb{R}$ to be a function that describes the consumption of player 1 after every history $h_t \in \mathcal{H}$. Feasibility requires

$$0 \leq c(h_{t-1}, L) \leq 1 \text{ and } 0 \leq c(h_{t-1}, H) \leq 1 + e \tag{F}$$

for all $h_{t-1} \in \mathcal{H}$. We will use c_{h_t} to denote the sharing rule induced by c after the history $h_t \in \mathcal{H}$. That is, for $h_t = (s_1, ..., s_t)$ and $h_k = (s'_1, ..., s'_k)$,

$$c_{h_t}(h_k) = c(h_t, h_k)$$

with (h_t, h_k) being defined as the history $(s_1, ..., s_t, s'_1, ..., s'_k)$. The sharing rule c_{h_0} after the null history is of course c. The sharing rule corresponding to aggregate endowment will be denoted ω, that is, $\omega(h_{t-1}, s_t) = \omega^{s_t}$ for all $h_{t-1} \in \mathcal{H}$.

Agents' expected utility from any sharing rule c can be written as

$$\begin{aligned} U^1(c) &= \sum_{t=1}^{\infty} \delta^{t-1} \sum_{h_t \in \mathcal{H}_t} u_1(c(h_t))p(h_t), \\ U^2(c) &= \sum_{t=1}^{\infty} \delta^{t-1} \sum_{h_t \in \mathcal{H}_t} u_2(\omega(h_t) - c(h_t))p(h_t). \end{aligned}$$

We will refer to a player's utility in period t conditional on $s_t = s$ as his interim utility. For any sharing rule c, agents' interim utility is

$$\begin{aligned} U_s^1(c) &= u_1(c(s)) + \delta U^1(c_s), \\ U_s^2(c) &= u_2(\omega(s) - c(s)) + \delta U^2(c_s). \end{aligned}$$

We can model a risk sharing agreement as a "gift exchange" game, as follows: agent 1 observes the state at the start of each period. He announces the state, and agents then simultaneously decide how much of their endowment to give to the other player. After transfers are made, the period ends; a new state is drawn and the process is repeated.

Note that any outcome path of the gift exchange game is a sharing rule. We are interested in sharing rules that correspond to outcome paths of subgame perfect Nash equilibria of the gift exchange game. We will call such sharing rules *sustainable.*

For a sharing rule c to correspond to a subgame perfect Nash equilibrium, the interim utility from $c_{h_{t-1}}$ must not be lower than that player's interim utility from a deviation at h_t. Let c^* be the sharing rule that corresponds to autarky, that is, $c^*(h_{t-1}, H) = e, c^*(h_{t-1}, L) = 0$ at all $h_{t-1} \in \mathcal{H}$. It is easy to see that c^* is the outcome path of a subgame perfect Nash equilibrium of the gift exchange game; in this equilibrium, players make zero transfers after every history. Moreover, no other subgame perfect Nash equilibrium of this game can give any player a worse payoff than the payoff associated with c^* after any history. Therefore, for a sharing rule c to be sustainable, c must satisfy the participation constraints

$$U_s^i(c_{h_{t-1}}) \geq U_s^i(c^*) \tag{P^i}$$

for all $h_{t-1} \in \mathcal{H}, s \in \{H, L\}$ and $i = 1, 2$. Additionally, it must be in agent 1's interest to always announce the state truthfully. As states are independent over time, we can without loss of generality state the revelation constraints as

$$U_H^1(c_{h_{t-1}}) \geq u_1(c_{h_{t-1}}(L) + e) + \delta U^1(c_{h'_t}) \tag{R^H}$$

for all $h_{t-1} \in \mathcal{H}$, with $h'_t = (h_{t-1}, L)$, and

$$U_L^1(c_{h_{t-1}}) \geq u_1(c_{h_{t-1}}(H) - e) + \delta U^1(c_{h''_t}) \tag{R^L}$$

for all $h_{t-1} \in \mathcal{H}$ such that $c_{h_{t-1}}(H) - e \geq 0$, and $h''_t = (h_{t-1}, H)$. If $c_{h_{t-1}}(H) - e < 0$ player 1 will of course not be able to pretend that the state is H, and this constraint can be ignored.

We can now give a characterization of sustainable sharing rules. To do this, consider the following strategy profile for the gift exchange game: given a sharing rule c, at any history $h_t = (h_{t-1}, s_t)$, player 1 announces the state truthfully. If $s_t = H$, player 1 makes the transfer $\max\{e - c_{h_{t-1}}(H), 0\}$, and player 2 makes the transfer $\max\{c_{h_{t-1}}(H) - e, 0\}$. If $s_t = L$, player 1 gives 0 to player 2, and player 2 gives him $c_{h_{t-1}}(L)$. A deviation by either player triggers reversion to autarky forever. This strategy profile induces the sharing rule c as an outcome path; moreover, if c satisfies P, R, it is easy to verify that the strategy profile constitutes a subgame perfect Nash equilibrium of the gift exchange game. Therefore, conditions P, R are necessary and sufficient for a sharing rule to be sustainable. We summarize the discussion above in the following definition:

Definition 1 *A sustainable sharing rule is a function* $c : \mathcal{H} \to \mathbb{R}$ *that satisfies* P, R *and* F *for all* $h_{t-1} \in \mathcal{H}$.

3 Efficient and sustainable sharing rules

In this section, we characterize efficient and sustainable sharing rules as solutions to a parametrized class of dynamic optimization problems.

Let $\mathcal{C}$ be the set of sustainable sharing rules. That is,

$$\mathcal{C} = \{c | c : \mathcal{H} \to \mathbb{R} \text{ and } c \text{ satisfies } P, R, F \text{ at all } h_{t-1} \in \mathcal{H}\}.$$

Definition 2 *A sharing rule* c *is efficient in the set* $\mathcal{D}$ *if* $c \in \mathcal{D}$ *and* c *is not Pareto dominated by another* $c' \in \mathcal{D}$. *The set of efficient sharing rules in* $\mathcal{D}$ *is denoted* $\mathcal{E}(\mathcal{D})$.

We are interested in the set $\mathcal{E}(\mathcal{C})$ of efficient and sustainable sharing rules. To simplify the characterization of sustainability, define $\mathcal{C}_0 \supset \mathcal{C}$ as

$$\mathcal{C}_0 = \{c | c : \mathcal{H} \to \mathbb{R} \text{ and } c \text{ satisfies } P^2, R^H, F \text{ at all } h_{t-1} \in \mathcal{H}\}.$$

Our first results concern sharing rules that are efficient in $\mathcal{C}_0$. In Proposition 1 below, we prove existence and uniqueness of such sharing rules. We also show that $\mathcal{E}(\mathcal{C}) = \mathcal{E}(\mathcal{C}_0)$, enabling us to ignore the constraints P^1 and R^L throughout the subsequent analysis. Consider the following maximization problem

$$W(\kappa) = \max U^2(c) \text{ subject to } c \in \mathcal{C}_0 \text{ and } U^1(c) \geq \kappa. \tag{2}$$

Let $S(\kappa)$ denote the set of sharing rules that solve the maximization problem above; that is, $c \in S(\kappa)$ if and only if $c \in \mathcal{C}_0, U^1(c) \geq \kappa$ and $U^2(c) = W(\kappa)$. Defining the set of player 1's attainable utility values as

$$\mathcal{K} = \{\kappa | \exists c \in \mathcal{C}_0 \text{ such that } U^1(c) = \kappa\},$$

it is easy to see that we can in fact write $\mathcal{E}(\mathcal{C}_0) = \{c | c \in S(\kappa) \text{ for } \kappa \in \mathcal{K}\}$. Proposition 1 also characterizes the set $\mathcal{K}$. Let $\kappa^* = U^1(c^*)$, and note that $\kappa^* = \min \mathcal{K}$. We define $\kappa^{**} = \sup \mathcal{K}$, and show that $\mathcal{K} = [\kappa^*, \kappa^{**}]$. Finally, we show that the function W defined in (2) is strictly decreasing and concave.

If $\mathcal{C}_0$ were convex, strict concavity of the u_i's would enable us to construct averages of two sharing rules $c', c'' \in \mathcal{C}_0$ that would give both players a strictly better than average payoff and still satisfy P^2, R^H and F. However, the sharing rule $\lambda c' + (1 - \lambda)c''$ may fail R^H even if both $c', c'' \in \mathcal{C}_0$. Our first result shows that it is nevertheless possible to construct a sharing rule $c \in \mathcal{C}_0$ from $c', c'' \in \mathcal{C}_0$ that yields exactly the average payoff to one agent, and a strictly higher payoff to the other. This result will play roughly the same role as convexity of $\mathcal{C}_0$ in our uniqueness and characterization result below.

Lemma 1 *For any $c', c'' \in \mathcal{C}_0$ such that $c' \neq c''$ and any $\lambda \in (0, 1)$, there exists $c \in \mathcal{C}_0$ such that*

$$\begin{aligned} U^1(c) &= \lambda U^1(c') + (1 - \lambda)U^1(c''), \\ U^2(c) &> \lambda U^2(c') + (1 - \lambda)U^2(c''). \end{aligned} \tag{3}$$

Proof: In the appendix.

With a convex constraint set and strictly concave utility functions, concavity of the value function of a maximization problem can easily be shown, as averages of optimal sharing rules are in the constraint set, and yield a strictly higher than average utility to both agents. (Uniqueness proofs generally follow along the same lines). Though our constraint set is not convex and we cannot take averages of sharing rules, Lemma 1 enables us to construct sharing rules that act like averages, in that they are guaranteed to be in the constraint set and yield at least average utility to both agents. Using Lemma 1, our proofs of uniqueness and concavity of the value function can then proceed in the usual fashion.

Proposition 1 *(i) $\mathcal{K} = [\kappa^*, \kappa^{**}]$; (ii) For $\kappa \in \mathcal{K}$, $S(\kappa)$ consists of a single element; (iii) $\mathcal{E}(\mathcal{C}) = \mathcal{E}(\mathcal{C}_0) = \{c | c \in S(\kappa) \text{ for } \kappa \in \mathcal{K}\}$; (iv) $W : \mathcal{K} \to \mathbb{R}$ is a strictly concave and strictly decreasing function.*

Proof: In the appendix.

Proposition 1 shows that $W(\kappa)$ has a unique solution, which we will denote by c^κ. It follows from the facts that $W(\kappa^{**}) = U^2(c^*)$ and W is strictly decreasing that if $\kappa^* < \kappa^{**}, c^*$ is not an efficient sharing rule; while if $\kappa^* = \kappa^{**}$, c^* is the only efficient and sustainable sharing rule. As not much remains to be said about the latter case, we shall henceforth assume $\kappa^* < \kappa^{**}$. Note that this is not a vacuous assumption; it is easy to show that for any collection of utility functions u_i satisfying our conditions, and for any $\pi \in (0, 1), e > 0$, we will have $\kappa^* < \kappa^{**}$ if δ is sufficiently close to 1.

The results in Proposition 1 enable us to state the problem $W(\kappa)$ in a more convenient dynamic programming form. Define, for all $\kappa \in \mathcal{K}$,

$$\begin{aligned} g(\kappa) &= c^\kappa(H), \\ G(\kappa) &= U^1(c^\kappa_H), \\ b(\kappa) &= c^\kappa(L), \\ B(\kappa) &= U^1(c^\kappa_L). \end{aligned}$$

The sharing rule c^κ thus yields player 1 consumption $g(\kappa)$ and a continuation utility of $G(\kappa)$ in his good state H; he obtains consumption $b(\kappa)$ and continuation utility $B(\kappa)$ in the bad state L.

As efficiency requires that the sharing rule c^κ be optimal at all t, h_t, we can write $W(\kappa)$ as

$$\begin{aligned} W(\kappa) = \max\, &\pi[u_2(1+e-g) + \delta W(G)] + (1-\pi)[u_2(1-b) + \delta W(B)] \\ &\text{subject to } \pi[u_1(g) + \delta G] + (1-\pi)[u_1(b) + \delta B] \geq \kappa, \\ &u_1(g) + \delta G \geq u_1(b+e) + \delta B, && (R^H) \\ &u_2(1+e-g) + \delta W(G) \geq U^2(c^*), && (P^2_H) \\ &u_2(1-b) + \delta W(B) \geq U^2(c^*), && (P^2_L) \\ &(g, b, G, B) \in [0, 1+e] \times [0, 1] \times \mathcal{K}^2. && (F') \end{aligned}$$

The functions $g : \mathcal{K} \to [0, 1+e]$, $b : \mathcal{K} \to [0, 1]$, $G : \mathcal{K} \to \mathcal{K}$ and $B : \mathcal{K} \to \mathcal{K}$ recursively determine c^κ for all values of $\kappa \in \mathcal{K}$. The sharing rule c^κ is fully characterized by the tuple $(\kappa,\, g,\, b,\, G,\, B)$: at h_0, $U^1(c^\kappa) = \kappa$. Consumption at $h_1 = s$ is by definition equal to $c^\kappa(s)$. In the next period, player 1's expected utility is $U^1(c^\kappa_s)$, and state dependent consumption is again determined by the functions g, b as either $g(U^1(c^\kappa_s)) = c^\kappa(s, H)$ or

$b(U^1(c^\kappa_H)) = c^\kappa(s, L)$. Continuing in this fashion, we see that (κ, g, b, G, B) is sufficient to determine $c^\kappa(h_t)$ at all t, h_t. For ease of notation, if the starting value κ is not relevant in the following, we will suppress the superscript κ and just write $c(h_t)$.

4 Excess Sensitivity and History Dependence

In this section, we utilize the dynamic programming characterization of efficient sharing rules to prove our main result. Proposition 2 below concerns efficiency of consumption allocations across states in an efficient and sustainable sharing rule. We show that risk sharing across states is never efficient in our economy.

Efficient risk sharing requires that agents' marginal rates of substitution are equalized across states. The function f associates with any bad state consumption b of agent 1 the corresponding good state consumption $f(b)$ that ensures the Pareto efficiency of the pair ($f(b)$, b). We show in Lemma 3 that nonincreasing absolute risk aversion implies differentiability of the u_i's, so that the marginal rates of substitution in the following are well defined.

Definition 3 *The graph of* $f : [0, 1] \to [0, 1+e]$ *describes efficiently insured consumption tuples:* $g = f(b)$ *if and only if*

$$\frac{u_1'(g)}{u_1'(b)} = \frac{u_2'(1+e-g)}{u_2'(1-b)}.$$

The strict concavity of the u_i's ensures that f is continuous and strictly increasing. In our example in the introduction, we had $f(b) = 2b$.

We define agents' consumption to exhibit *cross-sectional excess sensitivity* whenever $c_{h_{t-1}}(H) > f(c_{h_{t-1}}(L))$. Cross-sectional excess sensitivity is a necessary condition for any theory that tries to explain imperfections in the allocation of observed consumption risk. Limited commitment is often not enough to ensure this result. For example, in the economy with complete information and no commitment, consumption only displays cross-sectional excess sensitivity for intermediate values of δ. Our informational assumptions therefore ensure that markets are endogenously incomplete.

The following proposition shows that $c_{h_{t-1}}(H) = f(c_{h_{t-1}}(L))$ is never achievable in an efficient and sustainable sharing rule. To induce agent 1 to reveal the state correctly, his consumption in state H must be increased beyond the level implied by efficient risk sharing. Consumption thus displays cross-sectional excess sensitivity.

Proposition 2 *The informed agent always bears too much risk:*

$$g(\kappa) > f(b(\kappa)), \text{ for all } \kappa.$$

Proof: In the appendix.

The intuition behind this result is very simple. Efficient risk sharing in the presence of aggregate uncertainty requires that both agents absorb some risk, that is, $b < g < b+e$. As soon as agent 2 bears some risk, $g < b + e$, setting agent 1's continuation utilities equal across states would violate his revelation constraint. His incentives to reveal the state correctly must then be achieved by a spread in future consumption. If risk sharing is efficient today, the distortions from incentives are shifted entirely to the the future. However, as agents are risk averse, they prefer to distribute incentives more evenly over current and future consumption.

In the next section, we analyze the effects of asymmetric information on agents' ability to smooth consumption intertemporally in detail. To this end, define

$$\kappa^R := \max \kappa \in \mathcal{K} : U^1_H(c^\kappa) = u_1(b(\kappa) + e) + \delta B(\kappa).$$

The revelation constraint binds at all $\kappa \leq \kappa^R$ and is slack for $\kappa > \kappa^R$. Optimality implies $\kappa^R < \kappa^{**}$: recall that in autarky, agent 2 receives the same utility in both states. If he enters a risk sharing agreement with agent 1, his only source of uncertainty is agent 1's announcement of the state (note that agent 2 receives the same utility from agent 1 announcing state L correctly or falsely). As P^2 binds at κ^{**}, agent 2 is indifferent as to whether state H or L is announced, and is therefore also indifferent between agent 1 cheating or not. Optimality and uniqueness of $c^{\kappa^{**}}$ then require that in state H, the informed agent should strictly prefer the allocation $(g(\kappa^{**}), G(\kappa^{**}))$ to the alternative feasible allocation $(b(\kappa^{**})+e, B(\kappa^{**}))$. Therefore the revelation constraint is slack at κ^{**}. Hence, the presence of asymmetric information will only influence agents' consumption smoothing along history paths where agent 1's expected utility drops below κ^R.

Under complete markets, agents' consumption satisfies (i) $c_{h_{t-1}}(s_t) = c_{h_{t-1}}(s_t, s_{t+1})$ whenever $s_t = s_{t+1}$, and (ii) $f(c_{h_{t-1}}(L)) = c_{h_{t-1}}(L, H)$, $c_{h_{t-1}}(H) = f(c_{h_{t-1}}(H, L))$, for any $h_{t-1} \in \mathcal{H}$. If consumption satisfies both (i) and (ii), we refer to this as *efficient consumption smoothing*. Unlike cross-sectional excess sensitivity, efficient consumption smoothing is an intertemporal phenomenon, and therefore easily observable. Virtually all tests of the complete market hypothesis take the relations above as a starting point. The data show that consumption reacts more strongly to contemporaneous income shocks than predicted by a complete markets model (Cochrane (1996), Townsend (1994), Altonji et al. (1992), or Fafchamps and Lund (2003)). This effect is in general not large, but consumption is consistently found to vary too much with current income over time, suggesting a slow comovement of consumption with income. The fact that consumption seems to be imperfectly smoothed over time in the data suggests two distinct phenomena. Take a history where $s_t = H$. If $c_{h_{t-1}}(H) > f(c_{h_{t-1}}(H, L))$ is observed in the data, risk sharing across states cannot be efficient at (h_{t-1}, H), even if agents face the same optimization problem at (h_{t-1}, H) as at h_{t-1}. This phenomenon is the complete markets analogue of *excess sensitivity* to transitory income shocks as found by Hall and Mishkin (1982). Hall and Mishkin test the permanent income hypothesis, in which the agent's option is to self-insure, and find that transitory income shocks are not fully insured. These findings are similar enough in spirit to the evidence from tests of the complete markets hypothesis that we will in the following borrow their term to describe the overreaction of agents' consumption to transitory income shocks. In our model, we define agents' consumption to exhibit intertemporal excess sensitivity if

$$f(c_{h_{t-1}}(L)) < c_{h_{t-1}}(L, H) \text{ and } c_{h_{t-1}}(H) > f(c_{h_{t-1}}(H, L)).$$

We define state H to be agent 1's high-income state, state L to be agent 2's high-income state (as agent 2's share of aggregate income is larger in state L than in state H). Due to the aggregate resource constraint, our definition of intertemporal excess sensitivity encompasses both agents' responses to favorable and unfavorable income shocks.

The second phenomenon suggested by the data concerns history dependence of agents' consumption allocations. Again take a history where $s_t = H$. If $c_{h_{t-1}}(H) < c_{h_{t-1}}(H, H)$ is observed, it indicates that the agent does not face the same optimization problem at (h_{t-1}, H) than at h_{t-1}, even though the histories (h_{t-1}, H) and (h_{t-1}, H, H) have the same payoff-relevant consequences. We define agents' consumption to exhibit *intertemporal history dependence* if

$$c_{h_{t-1}}(H) < c_{h_{t-1}}(H, H) \text{ and } c_{h_{t-1}}(L) > c_{h_{t-1}}(L, L).$$

The next proposition shows that in an efficient and sustainable sharing rule, agent 1's consumption will display intertemporal excess sensitivity to all unfavorable income changes, that is, $c_{h_{t-1}}(H) > f(c_{h_{t-1}}(H, L))$ at all h_{t-1}. Furthermore, an agent who is exposed to successive favorable income shocks will experience strictly positive consumption growth, if the consumption process has not become trapped at his participation constraint (intertemporal history dependence).

Proposition 3 *In an efficient and sustainable sharing rule, an agent's consumption increases strictly during successive high-income states (intertemporal history dependence), that is,*

$$g(\kappa) < g(G(\kappa)) \text{ for } \kappa < \kappa^R, \text{ and } b(\kappa) > b(B(\kappa)) \text{ for } \kappa > \kappa^*.$$

Furthermore, agent 1's consumption always exhibits intertemporal excess sensitivity to an unfavorable income change:

$$g(\kappa) > f(b(G(\kappa))), \text{ for all } \kappa.$$

Proof: In the appendix.

After the first state transition, κ^R is the maximal attainable utility for agent 1. For all $\kappa \geq \kappa^R$, $G(\kappa) = \kappa$; if agent 1's expected utility is in $[\kappa^R, \kappa^{**}]$ and he is exposed to successive high income shocks, his consumption path will be constant over time. At κ^*, successive low income shocks similarly lead to flat consumption paths, as $B(\kappa^*) = \kappa^*$.

In an efficient and sustainable sharing rule, incentives are smoothed over present and future consumption. To induce agent 1 to announce the state correctly at (h_{t-1}, H), he must be rewarded with higher consumption $c_{h_{t-1}}(H)$ as well as better future prospects $U^1(c_{(h_{t-1},H)})$. This creates a ratchet effect: given that agent 1's expectations are higher after announcing H, he must be rewarded for announcing an additional high income shock by even higher consumption in state H. This is the intuition behind the positive consumption growth agent 1 experiences, as he is hit with successive favorable income shocks. A similar reasoning applies to successive unfavorable income shocks.

To illustrate the interaction between cross-sectional excess sensitivity and the intertemporal inefficiencies of Proposition 3, again suppose today's state s_t to be H, and that at $c_{h_{t-1}}(H)$, $c_{h_{t-1}}(H, H)$ and $c_{h_{t-1}}(H, L)$, no participation constraint binds. Consumption tomorrow displays cross-sectional excess sensitivity, that is, $c_{h_{t-1}}(H, H) > f(c_{h_{t-1}}(H, L))$. The smoothest allocation will have $c_{h_{t-1}}(H)$ lying strictly between $c_{h_{t-1}}(H, H)$ and $f(c_{h_{t-1}}(H, L))$. If participation constraints do not bind, today's as well as tomorrow's consumption can be moved freely without violating R^H, as long as agent 1 is kept indifferent to these changes. The smoothest allocation is then achievable for $s_t = H$. Therefore $c_{h_{t-1}}(H)$ will lie strictly between $c_{h_t}(H)$ and $f(c_{h_t}(L))$, for $h_t = (h_{t-1}, H)$. In the no commitment model with complete information, $c_{h_{t-1}}(H, H) > f(c_{h_{t-1}}(H, L))$ only if participation constraints bind, so that the smoothest allocation is not achievable; whereas if participation constraints do not bind at (h_{t-1}, H), $c_{h_{t-1}}(H, H) = f(c_{h_{t-1}}(H, L))$, so that efficient consumption smoothing is possible.

If today's state is L, lowering agent 1's consumption today makes the immediate consumption gain from cheating more attractive, even if expected future consumption is increased. Today's consumption can therefore not easily be decreased, so that $f(b(\kappa)) < g(B(\kappa))$ is not assured. However, $f(b(\kappa)) < g(B(\kappa))$ at $\kappa = \kappa^*$ and at $\kappa = \kappa^R$. Continuity of the solution to W implies that there exists $\kappa' > \kappa^*$, $\kappa'' < \kappa^R$ such that for all $\kappa \in [\kappa^*, \kappa')\cup\ (\kappa'', \kappa^P]$, this inequality remains strict. We show in Proposition 6 that agent 1's utility will be in the set $[\kappa^*, \kappa')\cup\ (\kappa'', \kappa^P]$ for a strictly positive fraction of time in the long run. Our computational results show $f(b(\kappa)) \leq g(B(\kappa))$ for all κ at all parameter values, however.

We therefore see that on most history paths, consumption reacts too strongly in response to an income shock, which is what is found in the data. In the no commitment economy with complete information, excess sensitivity is only present if agents are sufficiently impatient; intertemporal history dependence is always absent. In comparing the economy with binding participation constraints (intermediate δ) with the economy with asymmetric information, one now observes a very different pattern of consumption reactions to income changes. Let $(c_H,\ c_L)$ be an efficient contract under asymmetric information. To illustrate this point, we isolate an income change by looking at the following history: suppose in period t, the state has been L for a number of periods, and will be H for the next $k \geq 2$ periods. In the complete information economy, consumption immediately adjusts to this income change by jumping to c_H as soon as the state changes, and stays constant afterwards. Under asymmetric information, consumption might not react immediately (relative to aggregate income), if the history is such that $U^1(c_{h_t}) = \kappa$ and $f(b(\kappa)) = g(B(\kappa))$. However, it will then start to increase gradually towards the new income level. For the reverse income change from H to L, agent 1's consumption reacts immediately, but again decreases gradually afterwards. Consumption therefore reacts sometimes slowly, but always smoothly, to changes in individual income. Excess sensitivity under asymmetric information is therefore in stark contrast to the immediate and complete adjustment implied by excess sensitivity under complete information, and seems closer to the empirical evidence.[5]

In our model, constant consumption paths are then only observed if the consumption process is caught at the upper bound, $g(\kappa^R)$, or the lower bound, $b(\kappa^*)$. (For almost all parameter values, this is computed to occur less than 1% of the time in the long run in Section 6).

Cross-sectional excess sensitivity describes how an agent's consumption would change if his income today were different. The next proposition analyzes how an agent's consumption changes if a past income realization had been different. We refer to this as cross-sectional history dependence. To formally define cross-sectional history dependence, we define the following partial order on the set $\mathcal{H}$: given histories $h_t = (s_1, ..., s_t) \neq h'_t = (s'_1, ..., s'_t)$, h_t, $h'_t \in \mathcal{H}_t$, we will call the history h_t more fortunate than h'_t for a player if his high-income state occured in h'_t in any period $n \leq t$ where it also occured in h_t. That is, h_t is more fortunate for player 1 if $s'_n = H$ implies $s_n = H$, and the reverse is true for player 2. As h_t is more fortunate than h'_t for agent 1 if and only if h'_t is more fortunate than h_t for agent 2, we can state history dependence in terms of agent 1's consumption only. We say that consumption is *cross-sectionally history dependent* if $c(h_t) > c(h'_t)$, where h_t is more fortunate for agent 1 than h'_t. Proposition 4 shows that for all interior histories that do not get trapped at agents' participation constraints, consumption exhibits cross-sesctional history dependence.

Like cross-sectional excess sensitivity, cross-sectional history dependence is not easily observable, though the results in Altonji et al. (1992) suggest cross-sectional history dependence in risk sharing pools. Altonji et al. (1992) test altruism in extended families, a hypothesis that yields qualitatively similar predictions to efficient risk sharing. They concentrate not only on intertemporal consumption smoothing between adjacent periods, but also analyze the effects of individual incomes on agents' consumption over a longer time horizon. Though they believe that transfers within families exist, they reject altruism and risk sharing on the grounds that individual consumption displays

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[5] Under asymmetric information, an income change causes consumption growth for several periods. The slow reaction of consumption to income changes is also found in the empirical literature testing the permanent income hypothesis; for example Flavin (1981), or Zeldes (1989).

excess sensitivity to individual income with individual consumption, and that over time, an agent's position in the consumption distribution is correlated with his position in the distribution of individual incomes. We argue that, if incomes are imperfectly observed within the dynasty, Proposition 4 can help to explain those results.

Under complete markets, consumption smoothing is perfect, and history dependence always absent. In the no commitment model with complete information, the efficient contract is always characterized by a tuple (c_H, c_L) such that $c_{h_{t-1}}(H) = c_H$, $c_{h_{t-1}}(L) = c_L$ for all $h_{t-1} \in \mathcal{H}$. In a two-state world, consumption therefore depends only on current income for any history. In general, efficient contracts under complete information will exhibit perfect consumption smoothing wherever possible, and only depart from the first-best if an agent's participation constraint would be violated otherwise. In an environment with more states it is then possible that, as an agent's income changes, consumption changes to the Pareto efficient counterpart of the previous income level ($f(b)$ or $f^{-1}(g)$ in a two state model). Current consumption is then uniquely determined by current aggregate income and previous consumption, which was determined by previous income. Past income has a persistent effect on agents' consumption, until a participation constraint binds. But note that in the no commitment, complete information model, history dependence of consumption comes at the expense of excess sensitivity: previous income shocks have an effect on current consumption only if consumption has been perfectly smoothed since then. Under asymmetric information, excess sensitivity and history dependence coexist, as is found in the data.

The consumption process is bounded above and below by the presence of agents' participation constraints. An agent that is exposed to a long sequence of successive low income realizations will be driven down to his autarky utility. We state the next result for more balanced histories h_t, h_t' that do not get trapped at an agent's participation constraint (see the discussion following Proposition 5 for results at the boundaries). Note that this depends on the utility κ that agent 1 is started out with at $t = 1$. We therefore define the set $\mathcal{H}_t^\kappa \subseteq \mathcal{H}_t$ to be the set of histories of length t such that no agent's participation constraint binds at $\tau = 1, ..., t$ and $U^1(c) = \kappa$:

$$\mathcal{H}_t^\kappa = \{h_t \in \mathcal{H}_t | U^1(c_{h_\tau}^\kappa) > U^1(c^*) \text{ and } U_L^2(c_{h_\tau}^\kappa) > U_L^2(c^*), \tau \leq t\}.$$

To prove Proposition 4, we need to impose one of the following two assumptions on agent 1's utility:

Assumption C: u_1 exhibits constant absolute risk aversion.

Assumption L : u_1 is logarithmic: $u_1(c) = \ln(a + dc)$, $a \in \mathbb{R}_+$, $d > 0$.

If either C or L hold, we show in Proposition 4 that consumption displays cross-sectional history dependence. That is, for a history h_t that contains at least one favorable income shock for agent 1 at time τ, $s_\tau = H$, his consumption is strictly larger at h_t than if he had instead experienced an unfavorable income shock L at time τ. Favorable income shocks permanently increase an agent's consumption level. This result holds as long as agents do not become trapped at their participation constraints in either history.

Proposition 4 *(Cross-sectional history dependence) If either C or L hold and h_t is more fortunate for agent 1 than h'_t, his consumption is strictly higher at h_t than at h'_t :*

$$c^\kappa(h_t) > c^\kappa(h'_t),$$

for $h_t, h'_t \in \mathcal{H}^\kappa_t$ and an efficient and sustainable sharing rule c^κ, $\kappa \in \mathcal{K}$.

Proof: In the appendix.

Take a history h_t that only differs from h'_t at one $\tau \leq t$. To satisfy agent 1's revelation constraint at τ, either his present or future consumption in state H, or both, must be higher than in state L. As both agents' utility functions are concave, it is optimal to smoothe incentives over all present and future consumption, lowering future consumption after L while increasing it after H. However, the same is true for incentives at $\tau + k$. To get the unambiguous result of Proposition 4, we need to ensure that the downward tilt of agent 1's consumption profile after L is never stronger at $h_{\tau+k}$ than at $h'_{\tau+k}$. (This is equivalent to showing that the function B is monotone). We therefore need to impose more stringent assumptions on agent 1's utility function than have been necessary so far. Conditions C and L ensure that agent 1's marginal utility from cheating is not too different if his consumption level is increased, so that incentives are smoothed in a similar fashion along $h_{\tau+k}$ and $h'_{\tau+k}$.

As g, b and G are monotone away from agents' participation constraints and $G(\kappa) > B(\kappa)$ for all κ, a sharing rule will cause consumption to display cross-sectional history dependence if B is monotone. Conditions C and L are only sufficient to ensure monotonicity of B; we suspect this to be true for a wider range of environments as well. Indeed we find B monotone for CRRA utility and at all parameter values under study in our numerical examples in Section 6.

The following proposition states why the strong monotonicy of Proposition 4 can fail if agent 2's participation constraint binds. We show that there exists $\kappa^P \leq \kappa^R$ such that agent 2's participation constraint binds at all $\kappa \geq \kappa^P$. The function b that determines agent 1's low-state consumption for any value of κ is strictly decreasing on $[\kappa^P, \kappa^R]$: on this interval, a higher expected utility for agent 1 gives him strictly lower consumption in state L.

Proposition 5 *(Payment deferred) There exists κ^P, $\kappa^* < \kappa^P \leq \kappa^R$,such that the function b is strictly increasing on $[\kappa^*, \kappa^P]$, strictly decreasing on $[\kappa^P, \kappa^R]$. $\kappa^P < \kappa^R$ if and only if $B(\kappa^{**}) > \kappa^*$. Agent 2's participation constraint in the low state binds at all $\kappa \geq \kappa^P$.*

Proof: In the appendix.

The interval in which b is strictly decreasing is not void whenever $B(\kappa^{**}) > \kappa^*$. If $B(\kappa^{**}) = \kappa^*$, $B(\kappa) = \kappa^*$ for all κ, that is, any adverse income shock will throw agent 1 back to his autarky utility in the next period. There is then not much difference between best and worst outcomes in the risk sharing arrangement. If risk aversion and impatience take reasonable values, we do not expect this case to occur (indeed $B(\kappa^{**}) > \kappa^*$ for all parameter values in Section 6).

Agent 1's incentive to cheat comes from the immediate additional consumption he gains from a false announcement of state L. This immediate consumption gain is relatively more attractive, the smaller the transfer $c_{h_{t-1}}(L)$ from agent 1 is (for a given interim utility $U^1_L(c_{h_{t-1}})$). An efficient sharing rule will therefore usually force agent 1 to consume too much immediately in the low state and achieve incentives by lowering his future prospects $U^1(c_{(h_{t-1},L)})$, as long as this distortion is sustainable. Once agent 2's participation constraint in the low state binds, further distortions are no longer sustainable, and agent 1 can only be made better off by undoing the distortion, increasing his future prospects and decreasing current consumption. This translates into a region $[\kappa^P, \kappa^R]$ where the function b is strictly decreasing.

For two histories h_t, h_t' such that h_t is more fortunate for agent 1, the monotonicity result of Proposition 4 can then be turned around if agent 2's participation constraint binds at both h_t and h_t'. If agent 2 is held to his autarky utility at h_t', it is possible that $\kappa^R = U^1(c_{h_t}) = U^1(c_{h_t'})$, so that present and future consumption are identical at h_t and h_t'; in this case, the economy displays amnesia, in that the previous income histories cease to matter. However, if $\kappa^R > \kappa^P$, it is also possible that $\kappa^R > U^1(c_{h_t'})$, in which case $c_{h_{t-1}}(L) < c_{h_{t-1}'}(L)$ can be observed. Even though agent 2's participation constraint binds at both histories, the economy does not forget previous incomes, but instead promises higher consumption in the future for the more fortunate history h_t, at the expense of current consumption.

5 Long-run properties of the consumption process

This section adresses long-run wealth distributions of the agents. A wealth distribution in this context is understood to be a probability measure μ on $\mathcal{K}$. We show in Proposition 6 that the process $\{U^1(c_{h_t})\}_{t=1}^{\infty}$ has a unique stationary distribution $\bar{\mu}$ to which the process converges from any initial value. The ergodic set $\bar{\mathcal{K}}$, which forms the support of $\bar{\mu}$, is shown to span the absorbing set $[\kappa^*, \kappa^R]$, in the sense that the closure of $\bar{\mathcal{K}}$ contains both κ^* and κ^R. Therefore, in the long run, either agent may become impoverished, but neither agent will be impoverished forever. This result shows that fortunes may change, and the wealth distribution can change its composition. Furthermore we show that, if $B(\kappa^{**}) > \kappa^*$, the ergodic set also contains intermediate values of κ at which neither agent is impoverished.

Wealth distributions have been a focus of interest in the risk sharing literature for some time. Atkeson and Lucas (1992) and Thomas and Worrall (1990) analyze risk sharing in economies with adverse selection and full commitment. They show that the informed agents become impoverished in the long run. This is mainly due to the fact that they assume agents' utilities unbounded below, so that incentives can always be achieved by decreasing low-state consumption. The data do not seem to indicate such a unidirectional progression of inequality. In fact, Townsend (1994) finds that wealth and consumption distributions in the risk sharing pool can change composition comparatively freely within a generation. A result like Proposition 6 (ii), showing that agents' fortunes can change over time, is therefore desirable.

In the no commitment model with complete information, a very similar result holds true, as most agents' participation constraints will bind at some point in the long run. Agents will alternate being impoverished (in the sense that they only achieve their autarky utility), as the state changes. Fortunes therefore also change in the no commitment model. However, Ligon et al. (2002) find that the no commitment model with complete information predicts too high a degree of consumption inequality to fit the data. They calibrate the model to produce excess sensitivity that is close to the data. However, the no commitment model will only produce significant amounts of excess sensitivity if agents' participation constraints bind often, which may lead to a polarized wealth distribution. Proposition 6 (iii) shows that under asymmetric information, the consumption distribution can be more equitable, in the sense that there is a strictly positive fraction of time where neither agent is impoverished.

To make these notions more concise, let $\mathcal{B}(\mathcal{K})$ be the Borel σ-algebra on $\mathcal{K}$, so that $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$ is a measurable space, and let $M(\mathcal{K}, \mathcal{B}(\mathcal{K}))$ be the set of probability measures on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$. For any initial wealth distribution $\mu_0 \in M(\mathcal{K}, \mathcal{B}(\mathcal{K}))$, let $\mu_t(\mu_0)$ be the expected wealth distribution at t given the starting value μ_0. For the invariant measure $\bar{\mu}$, define $\bar{\mathcal{K}} = \{\kappa \in \mathcal{K} | \bar{\mu}(\kappa) > 0\}$.

Proposition 6 *(i) The process* $\{U^1(c_{h_t})\}_{t=1}^{\infty}$ *has a unique stationary distribution* $\bar{\mu}$*, and* $\mu_t(\mu_0)$ *converges to* $\bar{\mu}$ *at a uniform geometric rate, for any* $\mu_0 \in M(\mathcal{K}, \mathcal{B}(\mathcal{K}))$*;*

(ii) The ergodic set $\bar{\mathcal{K}}$ *is contained in* $[\kappa^*, \kappa^R]$*, and its closure contains both* κ^* *and* κ^R*;*

(iii) If $B(\kappa^{**}) > \kappa^*$*, there exists* $\kappa \in \bar{\mathcal{K}}$ *such that both agents are strictly better off than in autarky:* $\kappa \in (\kappa^*, \kappa^P)$*.*

Proof: In the appendix.

From any $U^1(c_{h_t})$, the utility process can visit κ^* in a finite number of periods; the stationary distribution $\bar{\mu}$ must therefore be unique. The revelation constraint will always bind after the first state transition. The set $(\kappa^R, \kappa^{**}]$, where R^H is slack, is therefore transient and cannot be contained in the ergodic set. The ergodic set always contains κ^*, as well as (at least) one point $\kappa \in [\kappa^P, \kappa^R]$ where agent 2's participation constraint binds.

6 Computation

In this section, we present some numerical results describing efficient and sustainable sharing rules. We first describe the algorithm used to derive the value function W and prove its convergence. Our algorithm makes use of the methods developed in Lemma 1 that act as our substitutes for convexity of the constraint set. For this reason, we have no need to use lotteries to convexify the constraint set as is done in much of the literature on numerical methods in informationally constrained dynamic principal-agent models (for example, Phelan and Townsend (1991)).

We define our algorithm as follows: let W_0 be the ex ante value function of the complete information, no commitment model in which agents have the same utility, discount factors and endowments as assumed in this paper. $\mathcal{K}_0$ is the set of attainable utility levels for agent 1 in this setting. Note that as we assume $\kappa^* < \kappa^{**}$ throughout, the complete information solution must necessarily differ from autarky as well. For $n = 1, 2, ...$, define the set $\mathcal{K}_n \in \mathbb{R}$ and the function $W_n : \mathcal{K}_n \to \mathbb{R}$ by

$$\begin{aligned} W_n(\kappa_n) = \max \pi[u_2(1+e-g) + \delta W_{n-1}(G)] &+ (1-\pi)[u_2(1-b) + \delta W_{n-1}(B)] \\ \text{subject to } \pi[u_1(g) + \delta G] + (1-\pi)[u_1(b) + \delta B] &\geq \kappa_n, \\ u_1(g) + \delta G \geq u_1(b+e) + \delta B, & \qquad (R^H) \\ u_2(1+e-g) + \delta W_{n-1}(G) \geq U^2(c^*), & \qquad (P_H^2) \\ u_2(1-b) + \delta W_{n-1}(B) \geq U^2(c^*), & \qquad (P_L^2) \\ (g, b, G, B) \in [0, 1+e] \times [0, 1] \times \mathcal{K}_{n-1}^2, \text{ and} & \qquad (F') \\ \mathcal{K}_n = [\kappa^*, \kappa_n^{**}], \text{ where } \kappa_n^{**} := \max \kappa : W_n(\kappa) \geq U^2(c^*). & \end{aligned}$$

Proposition 7 *$\{\mathcal{K}_n\}_{n=1}^{\infty}$ converges to $\mathcal{K}$, and the sequence of functions $\{W_n\}_{n=1}^{\infty}$ converges uniformly to W on $\{\mathcal{K}_n\}_{n=1}^{\infty}$.*

Proof: In the appendix.

To prove this result, we first adapt Lemma 1 to show that W_n is strictly concave for all n, which implies that the sequence $\{W_n\}$ is a sequence of continuous functions. It is intuitive that the sequence is also weakly decreasing, and that it is bounded below by W. We can therefore show that it converges uniformly to some continuous function $W_\infty \geq W$. To show that this limit must be equal to W, we make use of one Abreu, Pearce and Stacchetti's (1990) results, namely self-generation. By showing that the graph of the function W_∞ is a self-generating set, we can adapt their Theorem 1 to show that

$$graph W_\infty \subseteq \{(w^1, w^2) | \exists c \text{ s.t } w^1 = U^1(c), w^2 = U^2(c), \text{ and } c \text{ satisfies } P, R, F\},$$

the utility possibility set of sustainable sharing rules. As this set must be bounded above by $W \leq W_\infty$, we have the desired result.

To prove Proposition 4, we had to assume either logarithmic or CARA utility. For added generality, our numerical results have been derived assuming CRRA utility for both agents, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. For these utility functions, the Inada condition $\lim_{c\to 0} u'(c) = \infty$ holds, and $\frac{u_1'(e)}{u_1'(1)} \geq \frac{u_2'(1)}{u_2'(0)}$ is satisfied for any value of e, ruling out corner solutions (see footnote 2). Agent 1's income in the high state can then take any value. However, as agents' utility must be bounded in the relevant region, we can no longer assume that his income in state L is zero. We therefore assume a slightly different endowment process for agent 1 in this section. In particular, he receives $e^s > 0$ in state $s = H, L$, with $e^H > e^L$. All results are unchanged with this specification.

We set $\pi = 0.5$, $e^H = 1.9$, $e^L = 0.1$. Agent 2's sure income is again set to 1. This parametrization is chosen to match empirical research on entrepreneurial returns, which tend to be found equal on average to returns earned in less risky pursuits[6]. We assume identical utility functions for both agents, and derive our numerical results for varying degrees of impatience and risk aversion: $\sigma \in \{0.5, 2, 4\}$, $\delta \in \{0.9, 0.96, 0.99\}$.

Figure 1 presents policy functions for the problem W, selected to represent the greatest range of variation. As risk aversion and impatience become large, the policy functions become almost indistinguishable, and are represented in Figure 1 by $\sigma = 4$, $\delta = .99$.

The interval $[\kappa^P, \kappa^R]$ diminishes in size as σ and δ become large, but is visible in Figure 1 as the region where b is strictly decreasing (see Proposition 5). This region is almost too small to show up in Figure 1 for $\delta = .99$, $\sigma = 4$. The same is true for the interval $[\kappa^R, \kappa^{**}]$, where b is constant at $b(\kappa^{**})$.

The function B is strictly increasing on $[\max \kappa : B(\kappa) = \kappa^*, \kappa^R]$ for all parameter values. The monotonicity results in Proposition 4 therefore also apply to our parametriza-

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[6] Hamilton (2000) finds this in comparing wages earned by the self-employed versus the employed; Moskowitz and Vissing-Jorgensen (2002) regard the decision to become an entrepreneur as an investment decision and analyze returns of privately and publicly owned firms.

tion in this section: agent's consumption will display cross-sectional history dependence, within the bounds set by their participation constraints.

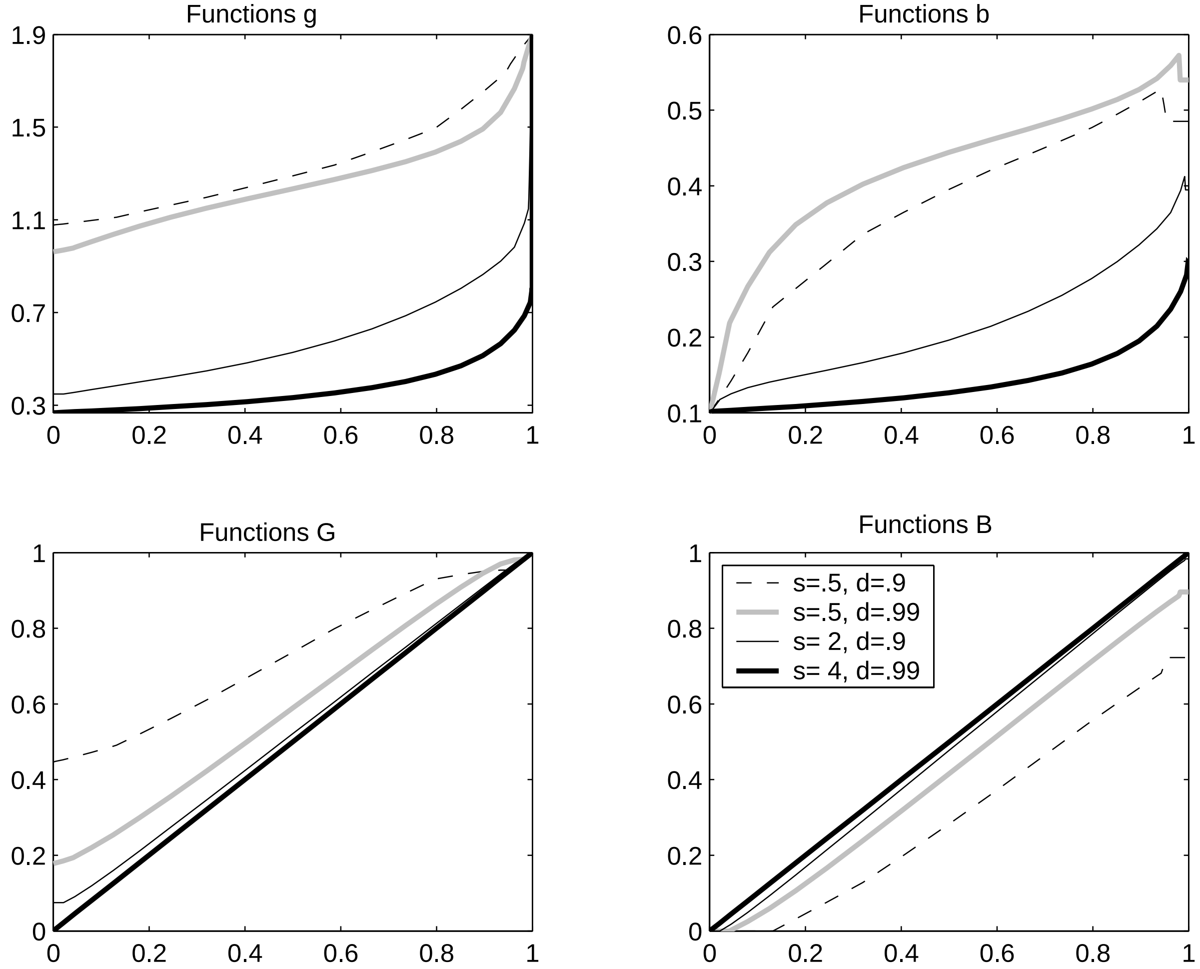

Figure 1: Selected policy functions (κ in percent)

Tables 1-3 show statistics describing the long-run properties of the consumption process. Expectations are taken with respect to the stationary distribution $\bar{\mu}$.[7]

In Table 1, we also provide the benchmark complete markets solution for comparison. For our parameter values, full insurance is possible in the no commitment, complete information problem. Consumption statistics for this environment will therefore be similar to the complete markets solution. Average consumption under asymmetric information does not seem to differ greatly from the benchmark, especially as players become more patient. The fact that expected consumption for $\sigma = 4$, $\delta = .99$ seems to be unexpectedly high.is very likely a consequence of our approximation to $\bar{\mu}$. As patience and risk aversion become large, the functions G and B approach the diagonal very closely (see Figure 1). The transition matrix used to approximate $\bar{\mu}$ then has many eigenvalues very close to one, which makes the decomposition routines (and therefore the resulting eigenvector) less robust.

$\sigma \diagdown \delta$	.9	.96	.99	complete markets
.5	1.3804 (0.3933)	1.2849 (0.4508)	1.2481 (0.4681)	1.2763 (0.4841)
2	0.5374 (0.1989)	0.5021 (0.1887)	0.4315 (0.1633)	0.7136 (0.2707)
4	0.2874 (0.1088)	0.2805 (0.1063)	0.3695 (0.1401)	0.3483 (0.1321)

Table 1: Agent 1's expected consumption in state $H(L)$

However, Table 2 shows one of the main differences of the consumption process with asymmetric information to a complete information solution. Covariances of consumption with lagged income are strictly positive. If markets were complete, these covariances would be zero for any lag strictly greater than zero. These covariances are usually positive in the data, and decrease in magnitude with the length of the lag. Table 2 shows that our model makes qualitatively the same predictions.

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[7] We have discretized $\mathcal{K}$, using a grid of roughly 1000 points, to derive a transition matrix for the Markov utility process, using functions G and B. The eigenvector corresponding to the unity eigenvalue of this transition matrix is our approximation of $\bar{\mu}$.

$\sigma\diagdown\delta$	.9	.96	.99
.5	0.4442	0.3753	0.3510
	0.0987	0.0352	0.0057
	0.0709	0.0321	0.0056
	0.0478	0.0291	0.0055
2	0.1523	0.1410	0.1207
	0.0076	0.0028	0.0006
	0.0074	0.0027	0.0006
	0.0073	0.0027	0.0006
4	0.0804	0.0784	0.1032
	0.0003	0.0001	0.0003
	0.0003	0.0001	0.0003
	0.0003	0.0001	0.0003

Table 2: Consumption/income covariances (income is lagged k periods, $k = 0, ...4$)

We have shown in Proposition 6 that no agent's utility converges to his autarky utility in the long run, but that both agents' participation constraints will bind at times. Table 3 shows that there is also a significant amount of time where no agent's participation constraint binds. In fact, unlike in the literature that assumes full commitment, both agents are simultaneously better off than in autarky for at least 83 percent of the time (and in most cases, for more than 99 percent of the time). In the context of Proposition 3, this implies that the constant consumption paths that occur when agents are trapped at a participation constraint can only be observed less than 83 percent of the time. Similarly, it will also not be hard to find histories that fulfill the requirements of Proposition 4, that is, histories that do not cause consumption to touch a selected boundary.

$\sigma\diagdown\delta$	.9	.96	.99
.5	5.49 (10.65)	0.68 (1.48)	0.03 (0.07)
2	0.543 (0.014)	0.116 (0.0037)	0.015 (0.002)
4	0.198 (1.12e-009)	0.077 (9.8e-007)	0.05 (0.177)

Table 3: Time spent at agent 1's autarky utility ($\bar{\mu}(\kappa^*)$)in the long run (in percent), in parentheses: time spent at agent 2's autarky utility ($\bar{\mu}([\kappa^P, \kappa^R])$).

Table 3 also indicates how varying risk aversion and impatience influences the stationary distribution. We further illustrate this trend in Figure 2, where we plot selected wealth distributions $\bar{\mu}$. Increasing risk aversion and impatience shifts the wealth distribution to the left, towards agent 1's autarky utility. This is again a consequence of the downward distortion of agent 1's low-state consumption profile. Recall that agent 1 can be kept indifferent by increasing his current low-state consumption while decreasing his continuation utility after state L, and that this change strictly diminishes his incentive to cheat as u^1 is concave. Agent 1's continuation utility will have to be decreased by a larger amount if he is more impatient, leading to a stronger downward tilt. On the other hand, his utility of cheating $u^1(b(\kappa)+e)$ will react more strongly compared to his utility of current consumption $u^1(b(\kappa))$, the more risk averse he is (if agent 1 were risk neutral, there could be no downward tilt at all). Increasing risk aversion therefore also serves to increase the downward tilt, shifting the wealth distribution to the left.

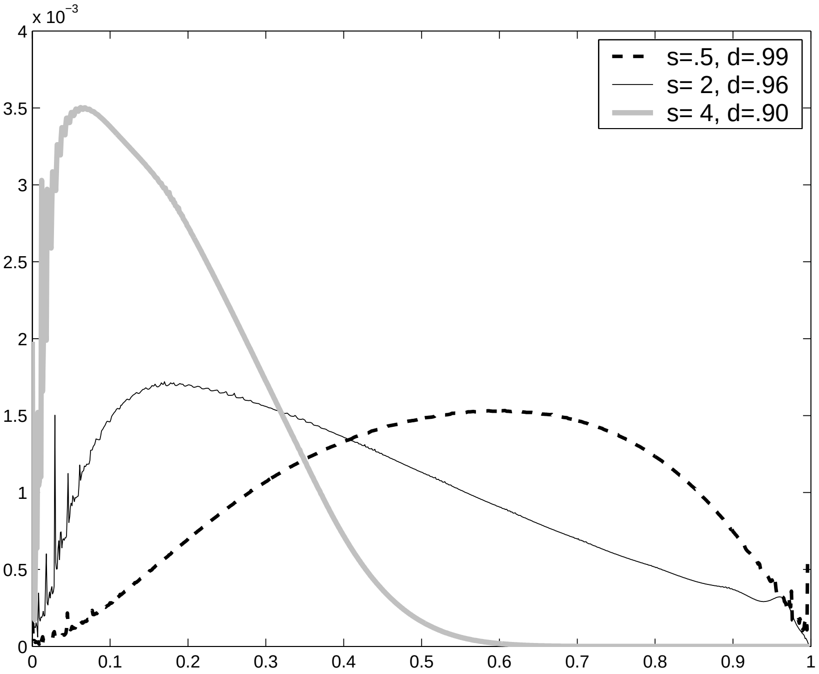

Figure 2: Selected stationary distributions $\bar{\mu}$ (κ in percent)

7 Conclusion

In this paper, we have argued that the presence of asymmetric information gives rise to excess sensitivity and makes consumption history dependent, in a way that has been widely documented in the data. Information problems will in general serve to make allocations history dependent, as well as allocating too much risk to the informed agent and thus ensuring that consumption is not efficiently insured. In fact, informational asymmetries have for some time been proposed as possible explanations in empirical work documenting consumption inefficiencies (for example Townsend (1994), or Ligon (1998)). Our goal in this paper has been to demonstrate that this is indeed the case. We have used a very simple exchange economy to make these points. In particular, we have assumed that agent 1's income can only take two values. However, we do not expect that a more general income process would change the substance of our results.

Throughout the paper, we have described a relatively symmetric risk sharing situation, where agents have similar incomes, both agents are risk averse, and neither can commit. This specification seems most appropriate for risk sharing between equals, such as in social networks or between sovereign states. However, our model could also be extended to cover other applications, such as unemployment insurance or formal credit contracts, in which the uninformed agent can commit and/ or is risk neutral. If the uninformed agent were risk neutral, our results would be entirely unchanged. The symmetric no commitment assumption puts lower and upper boundaries on the consumption process through agents' participation constraints. Our results on excess sensitivity and history dependence of consumption are stated for interior histories where no agent's participation constraints bind. The removal of an agent's participation constraint (for example, by letting the uninformed agent commit) will therefore not change these results (Propositions 1-4); Proposition 5 describes the process at agent 2's participation constraint. The behavior of the consumption process in the long run (Proposition 6) is, however, affected by the presence of agents' participation constraints. Whether a stationary wealth distribution exists if the uninformed agent can commit will depend on the particulars of the problem, though there are possible cases in which the informed agent's utility might end up below his autarky utility forever. In this case, a focus on sustainable policies as analyzed in this paper might be desirable.

8 Appendix

8.1 Proof of Proposition 1

Proof of Lemma 1

To construct c from $c', c'' \in \mathcal{C}_0$, choose $c(h_t)$ such that

$$u_1(c(h_t)) = \lambda u_1(c'(h_t)) + (1-\lambda)u_1(c''(h_t))$$

for all $h_t \in \mathcal{H}$. It follows from the concavity (and hence continuity) of u_1 that such a $c(h_t)$ can be chosen. As u_1 is concave, we conclude $c(h_t) \geq \lambda c'(h_t) + (1-\lambda)c''(h_t)$, which implies by strict concavity of u_2 that

$$u_2(\omega(h_t) - c(h_t)) \geq \lambda u_2(\omega(h_t) - c'(h_t)) + (1-\lambda)u_2(\omega(h_t) - c''(h_t)),$$

with a strict inequality whenever $c'(h_t) \neq c''(h_t)$. But such a h_t must exist as $c' \neq c''$. Therefore, c satisfies equation 3 and P^2. To conclude the proof we need to show that c also satisfies R^H. The argument below is valid at any history; it will be sufficient to consider the situation at $t = 1$. By assumption,

$$\begin{aligned} u_1(c'(H)) + \delta U^1(c'_H) &\geq u_1(c'(L) + e) + \delta U^1(c'_L) \\ u_1(c''(H)) + \delta U^1(c''_H) &\geq u_1(c''(L) + e) + \delta U^1(c''_L) \end{aligned}$$

Then

$$\begin{aligned} u_1(c(H)) + \delta U^1(c_H) &= \lambda[u_1(c'(H)) + \delta U^1(c'_H)] + (1-\lambda)[u_1(c''(H)) + \delta U^1(c''_H)] \\ &\geq \lambda[u_1(c'(L) + e) + \delta U^1(c'_L)] + (1-\lambda)[u_1(c''(L) + e) + \delta U^1(c''_L)] \\ &= \lambda u_1(c'(L) + e) + (1-\lambda)u_1(c''(L) + e) + \delta U^1(c_L). \end{aligned}$$

Since u_1 exhibits nonincreasing absolute risk aversion and by definition $u_1(c(L)) \leq \lambda u_1(c'(L)) + (1-\lambda)u_1(c''(L))$, we can conclude

$$u_1(c(L) + e) \leq \lambda u_1(c'(L) + e) + (1-\lambda)u_1(c''(L) + e).$$

Therefore,

$$u_1(c(H)) + \delta U^1(c_H) \geq u_1(c(L) + e) + \delta U^1(c_L),$$

as desired. ■

We first prove parts (i)-(iii) of Proposition 1 in Lemma 2. Part (iv) requires characterization results from the proof of Proposition 2. We therefore solve the relaxed problem in Proposition 2 and show that it is equivalent to the full problem in Lemma 14, which then concludes the proof of Proposition 1.

Lemma 2 *(i)* $\mathcal{K} = [\kappa^*, \kappa^{**}]$; *(ii)* $S(\kappa)$ *consists of a single element; (iii)* $W : \mathcal{K} \to \mathbb{R}$ *is a strictly concave and strictly decreasing function.*

Proof. (i) $\mathcal{C}_0 \subset B(\mathcal{H}, \mathbb{R})$, the set of bounded functions $c : \mathcal{H} \to \mathbb{R}$. We endow this space with the topology of pointwise convergence. Then U^1, U^2 are continuous under this topology, and $\mathcal{C}_0$ is compact. Therefore, the maximum $\kappa^{**} = \max_{c \in \mathcal{C}_0} U^1(c)$ subject to $U^2(c) \geq U^2(c^*)$ exists, and $\kappa^{**} = \max \mathcal{K}$. Then, for all $\kappa = \lambda\kappa^* + (1-\lambda)\kappa^{**}$ there exists $c \in \mathcal{C}_0$ such that $U^1(c) = \kappa$ by Lemma 1, for all $\lambda \in (0,1)$.

To prove (ii), we first show that $S(\kappa) \neq \varnothing$. As in (i), U^1, U^2 are continuous under the topology of pointwise convergence, and the set $\{c | c \in \mathcal{C}_0 \text{ and } U^1(c) = \kappa, \text{ for } \kappa \in \mathcal{K}\}$ is compact. Therefore, the maximum $W(\kappa)$ is attained. To show uniqueness of the solution, suppose there exist $c', c'' \in S(\kappa)$ with $c' \neq c''$. By Lemma 1, there exists $c \in \mathcal{C}_0$ with the properties that $U^1(c) = \kappa$ and $U^2(c) > W(\kappa)$, which is a contradiction.

(iii) To prove strict concavity of W, we take $c' \in S(\kappa')$ and $c'' \in S(\kappa'')$ such that $\kappa' \neq \kappa''$. By Lemma 1, for any $\lambda \in (0,1)$, there exists $c \in \mathcal{C}_0$ such that

$$U^1(c) = \lambda\kappa' + (1-\lambda)\kappa'',$$

and therefore

$$W(\lambda\kappa' + (1-\lambda)\kappa'') \geq U^2(c) > \lambda W(\kappa') + (1-\lambda)W(\kappa'').$$

To show strict decreasingness of W, consider $\kappa, \kappa' \in \mathcal{K}$ with $\kappa < \kappa', c = S(\kappa), c' = S(\kappa')$. Note that c' satisfies $c' \in \mathcal{C}_0$ and $U^1(c') \geq \kappa$. As $U^1(c) = \kappa \neq \kappa' = U^1(c')$, $W(\kappa) = W(\kappa')$ contradicts uniqueness of of $S(\kappa)$, while $W(\kappa) < W(\kappa')$ contradicts $c = S(\kappa)$. Therefore $W(\kappa) > W(\kappa')$. ■

8.2 Proof of Proposition 2

We first show that agents' marginal rates of substitution are well-defined:

Lemma 3 $u_i : \mathbb{R} \to \mathbb{R}$ *is everywhere differentiable,* $i \in \{1, 2\}$.

Proof. For any $c \in \mathbb{R}$ and $\varepsilon > 0$, we can define

$$p^i(c,\varepsilon) = \frac{u_i(c) - u_i(c-\varepsilon)}{[u_i(c) - u_i(c-\varepsilon)] + [u_i(c+\varepsilon) - u_i(c)]}.$$

Note that

$$\begin{aligned} \lim_{\varepsilon\to 0} p^i(c,\varepsilon) &= \lim_{\varepsilon\to 0} \frac{u_i(c) - u_i(c-\varepsilon)}{[u_i(c) - u_i(c-\varepsilon)] + [u_i(c+\varepsilon) - u_i(c)]} = \\ &= \lim_{\varepsilon\to 0} \frac{[u_i(c) - u_i(c-\varepsilon)]/\varepsilon}{[u_i(c) - u_i(c-\varepsilon)]/\varepsilon + [u_i(c+\varepsilon) - u_i(c)]/\varepsilon}. \end{aligned}$$

As u_i is strictly concave, the limits

$$\lim_{\varepsilon\to 0}[u_i(c) - u_i(c-\varepsilon)]/\varepsilon, \lim_{\varepsilon\to 0}[u_i(c+\varepsilon) - u_i(c)]/\varepsilon$$

exist, and we can conclude

$$\lim_{\varepsilon\to 0} p^i(c,\varepsilon) = \frac{(u_i)'_-(c)}{(u_i)'_-(c) + (u_i)'_+(c)}.$$

At any point c where u_i is differentiable, $(u_i)'_-(c) = (u_i)'_+(c)$ and therefore $\lim_{\varepsilon\to 0} p^i(c,\varepsilon) = \frac{1}{2}$; at all other points c' strict concavity of u_i implies $(u_i)'_-(c') > (u_i)'_+(c')$, and hence $\lim_{\varepsilon\to 0} p^i(c',\varepsilon) = k > \frac{1}{2}$. By strict concavity of u_i, we know that u_i is a.e. differentiable and that, for any c' where u_i is not differentiable, we can therefore find $c < c'$ such that u_i is differentiable at c. As $p^i(c,\varepsilon) \to \frac{1}{2}$ and $p^i(c',\varepsilon) \to k > \frac{1}{2}$ and p^i is continuous due to strict concavity of u^i, there exists $\varepsilon' > 0$ such that $p^i(c,\varepsilon') < p^i(c',\varepsilon')$. But by definition of $p^i(c,\varepsilon)$,

$$u_i(c) = p^i(c,\varepsilon)u_i(c+\varepsilon) + (1 - p^i(c,\varepsilon))u_i(c-\varepsilon)$$

and therefore if $c < c'$, nonincreasing absolute risk aversion implies

$$u_i(c') \leq p^i(c,\varepsilon)u_i(c'+\varepsilon) + (1 - p^i(c,\varepsilon))u_i(c'-\varepsilon),$$

and we conclude that $p^i(c,\varepsilon) \geq p^i(c',\varepsilon)$ for any $\varepsilon > 0$. Therefore u_i must be differentiable everywhere. ■

We will show that $\frac{u_1'(g(\kappa))}{u_1'(b(\kappa))} \geq \frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))}$ is not efficient by identifying a pareto improving change in the allocation. To make sure these changes are feasible, the next lemmas serve to identify the regions of $\mathcal{K}$ where R^H, P^2 and F bind.

Lemma 4 *(i) If P^2_H is slack at $\kappa, G(\kappa) \geq B(\kappa)$; if also $b(\kappa) > 0$,*

$$\frac{u_1'(g(\kappa))}{u_1'(b(\kappa))} \leq \frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))};$$

(ii) P^2_L and R^H cannot both be slack at some κ.

Proof. To show (i), suppose $G(\kappa) < B(\kappa)$ at some κ such that P_H^2 is slack. Then a change $G(\kappa) + \Delta, B(\kappa) - k\Delta$ is feasible for small Δ and $k = \frac{\pi}{1-\pi}$, and constitutes a Pareto improvement, while still satisfying R^H and P_L^2. Similarly, suppose

$$\frac{u_1'(g(\kappa))}{u_1'(b(\kappa))} > \frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))}$$

(note that this implies $g < 1 + e$): then any change $g(\kappa) + \Delta, b(\kappa) - k\Delta$, with

$$k \in (\frac{\pi}{1-\pi}\frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))}, \frac{\pi}{1-\pi}\frac{u_1'(g(\kappa))}{u_1'(b(\kappa))}),$$

still satisfies R^H and P_L^2 as well as P_H^2 for small enough Δ and constitutes a Pareto improvement. It is feasible if $b(\kappa) > 0$.

To show (ii), suppose at some κ both R^H and P_L^2 are slack, and $b(\kappa) < 1$. If

$$\frac{u_1'(g(\kappa))}{u_1'(b(\kappa))} < \frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))},$$

(note that this implies $g > 0$), a change $g(\kappa) - \Delta, b(\kappa) + k\Delta$ is feasible and satisfies R^H, P^2 for small Δ and

$$k \in \left(\frac{\pi}{1-\pi}\frac{u_1'(g(\kappa))}{u_1'(b(\kappa))}, \frac{\pi}{1-\pi}\frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))}\right),$$

and constitutes a Pareto improvement. Therefore,

$$\frac{u_1'(g(\kappa))}{u_1'(b(\kappa))} \geq \frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))},$$

which implies $g(\kappa) \leq b(\kappa) + e$. If $b(\kappa) = 1$, $g(\kappa) \leq b(\kappa) + e$ also. Additionally, if $G(\kappa) > B(\kappa)$, a change $G(\kappa) - \Delta, B(\kappa) + k\Delta$ is feasible for small Δ and $k = \frac{\pi}{1-\pi}$, and constitutes a Pareto improvement. R^H and P_L^2 slack therefore imply $G(\kappa) \leq B(\kappa)$. But if $g(\kappa) \leq b(\kappa) + e$ and $G(\kappa) \leq B(\kappa)$, R^H is either tight or violated, which is a contradiction. ■

Definition 4 *For $\Xi \subset \mathbb{R}$ compact and convex, $\bar{x} \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$, define the maximization problem $V(\xi; f, \bar{x})$ as*

$$\begin{aligned} V(\xi; f, \bar{x}) = \max u_2(\bar{x} - x) + \delta W(X) \text{ subject to} \\ f(x) + \delta X \geq \xi, \xi \in \Xi \\ (x, X) \in [0, \bar{x}] \times \mathcal{K}. \end{aligned}$$

The solutions to $V(\xi; f, \bar{x})$ are denoted $x(\xi; f, \bar{x}), X(\xi; f, \bar{x})$.

Lemma 5 *If f is strictly increasing, strictly concave and C^1, x, X are continuous functions. Moreover, $X(\cdot; f, \bar{x})$ is weakly increasing for all $\xi \in \Xi$, and $x(\cdot; f, \bar{x})$ is strictly increasing for all $\xi \in \Xi$ such that $x(\xi) \in (0, \bar{x})$.*

Proof. By concavity of f, u_2, W and convexity of the constraint set, the solutions to the problem above are continuous functions. For $\xi \in \Xi$, $x = x(\xi; f, \bar{x})$, $X = X(\xi; f, \bar{x})$ must satisfy the following first-order conditions:

$$\begin{aligned}\frac{u_2'(\bar{x}-x)}{f'(x)} &\geq -v_X, \text{ if } x < \bar{x} \text{ and } X > \kappa^*,\\ \frac{u_2'(\bar{x}-x)}{f'(x)} &\leq -v_X, \text{ if } x > 0 \text{ and } X < \kappa^{**},\end{aligned}$$

with $v_X \in \partial W(X)$ (the subdifferential of W always exists due to strict concavity of W). Let $\xi' > \xi''$, $\xi', \xi'' \in \Xi$, and define $x' = x(\xi'; f, \bar{x})$, $X' = X(\xi'; f, \bar{x})$, $x'' = x(\xi''; f, \bar{x})$, $X'' = X(\xi''; f, \bar{x})$. Then

$$\xi' = f(x') + \delta X', \xi'' = f(x'') + \delta X'',$$

and either $x' > x''$ or $X'' > X''$. Suppose $\bar{x} \geq x' > x'' \geq 0$. If $X' = \kappa^{**}$ or $X'' = \kappa^*, X' \geq X''$. If $X' < \kappa^{**}$ and $X'' > \kappa^*$,

$$-v_{X'} \geq \frac{u_2'(\bar{x}-x')}{f'(x')} > \frac{u_2'(\bar{x}-x'')}{f'(x'')} \geq -v_{X''}, \text{ with } v_{X'} \in \partial W(X'), v_{X''} \in \partial W(X''),$$

and strict concavity of W then implies that $X' \geq X''$. It remains to be shown that $x' > x''$ whenever $\kappa^{**} \geq X' > X'' \geq \kappa^*$ and $x' < \bar{x}$, $x'' > 0$. Then,

$$\frac{u_2'(\bar{x}-x')}{f'(x')} \geq -v_{X'} > -v_{X''} \geq \frac{u_2'(\bar{x}-x'')}{f'(x'')},$$

and therefore $x' > x''$. Hence, $x' > x''$ unless $x' = x'' = 0$ or $x' = x'' = \bar{x}$. ■

Definition 5 $\gamma(\kappa) = u_1(g(\kappa)) + \delta G(\kappa) = U_H^1(c^\kappa)$,
$\beta(\kappa) = u_1(b(\kappa)) + \delta B(\kappa) = U_L^1(c^\kappa)$.

Claim 1 $g(\kappa) = g(\gamma(\kappa); u_1, 1+e)$, $G(\kappa) = G(\gamma(\kappa); u_1, 1+e)$, *for all* κ.

Proof. Suppose not. Then $(g', G') := (g(\gamma(\kappa); u_1, 1+e),\ G(\gamma(\kappa); u_1, 1+e))$ satisfy $u_1(g') + \delta G' = \gamma(\kappa)$. R^H and the adding up constraint therefore hold. Furthermore, by definition of (g', G'),

$$u_2(1+e-g') + \delta W(G') > u_2(1+e-g(\kappa)) + \delta W(G(\kappa)),$$

satisfying P_H^2 and contradicting optimality of $(g(\kappa), G(\kappa))$. ■

Claim 2 *$b(\kappa) \geq b(\beta(\kappa); u_1, 1)$, $B(\kappa) \leq B(\beta(\kappa); u_1, 1)$, for all κ; with equality whenever R^H is slack.*

Proof. We first show that the weak inequality must always hold. Suppose not, and again denote $(b', B') := (b(\beta(\kappa); u_1, 1+e),\ B(\beta(\kappa); u_1, 1+e))$. As $u_1(b') + \delta B' = \beta(\kappa) = u_1(b(\kappa)) + \delta B(\kappa)$, this implies $b' > b(\kappa), B' < B(\kappa)$. By definition of (b', B'),

$$u_2(1 - b') + \delta W(B') > u_2(1 - b(\kappa)) + \delta W(B(\kappa)).$$

b', B' therefore satisfy P_L^2. To verify that b', B' satisfy R^H, note that

$$\delta(B(\kappa) - B') = u_1(b') - u_1(b(\kappa)) \geq u_1(b' + e) - u_1(b(\kappa) + e)$$

by concavity of u_1, and therefore

$$u_1(g(\kappa)) + \delta G(\kappa) \geq u_1(b(\kappa) + e) + \delta B(\kappa) \geq u_1(b' + e) + \delta B',$$

contradicting optimality of $b(\kappa), B(\kappa)$. If R^H is slack and $b(\kappa) > b', B(\kappa) < B'$, a change $b(\kappa) - \Delta, B(\kappa) + \Delta^B$ such that

$$u_1(b(\kappa) - \Delta) + \delta(B(\kappa) + \Delta^B) = u_1(b') + \delta B'$$

still satisfies R^H for Δ small enough; strict concavity of u_i, W implies

$$u_2(1 - (b(\kappa) - \Delta)) + \delta W(B(\kappa) + \Delta^B) > u_2(1 + e - b(\kappa)) + \delta W(B(\kappa)),$$

satisfying P_L^2 and contradicting optimality of $(b(\kappa), B(\kappa))$. ■

Lemma 6 *There exists $\kappa^R \in \mathcal{K}$ such that P_L^2 binds for all $\kappa \geq \kappa^R$, and R^H binds for all $\kappa \leq \kappa^R$ (and is slack elsewhere).*

Proof. Step 1: Either $G(\kappa^{**}) = B(\kappa^{**}) < \kappa^{**}$, or R^H is slack at κ^{**}.

Proof of Step 1: Take the maximization problem

$$\max \pi[u_1(g) + \delta G] + (1 - \pi)[u_1(b) + \delta B] \text{ subject to}$$

$$u_2(1 + e - g) + \delta W(G) \geq u_2(1) + \delta U^2(c^*), \tag{P_H^2}$$

$$u_2(1 - b) + \delta W(B) \geq u_2(1) + \delta U^2(c^*), \tag{P_L^2}$$

$$(g, b, G, B) \in [0, 1+e] \times [0, 1] \times \mathcal{K}^2,$$

which is equivalent to the problem $W(\kappa^{**})$ without R^H. Let the (unique) solution to this problem be denoted (g^{**}, b^{**}, G^{**}, B^{**}). If this solution satisfies R^H, it must be equal to ($g(\kappa^{**})$, $b(\kappa^{**})$, $G(\kappa^{**})$, $B(\kappa^{**})$). First suppose that $B^{**} > G^{**}$. From P^2, we see that then $g^{**} > e$ and $b^{**} < g^{**} - e \leq 1$. Then there exist $v_B \in \partial W(B^{**}), v_G \in \partial W(G^{**})$ such that

$$\begin{aligned} u_1'(g^{**}) &\geq \frac{u_2'(1+e-g^{**})}{-v_G}, \\ u_1'(b^{**}) &\leq \frac{u_2'(1-b^{**})}{-v_B}. \end{aligned}$$

But $B^{**} > G^{**}$ implies $-v_B > -v_G$, and therefore

$$\frac{u_2'(1+e-g^{**})}{u_1'(g^{**})} < \frac{u_2'(1-b^{**})}{u_1'(b^{**})},$$

which contradicts $b^{**} < g^{**} - e$. Therefore, we must have $B^{**} \leq G^{**}$. From P^2, we see that if $G^{**} = B^{**} = \kappa^{**}$, $b^{**} = 0$ and $g^{**} = e$, and therefore

$$\pi[u_1(g^{**}) + \delta G^{**}] + (1-\pi)[u_1(b^{**}) + \delta B^{**}] = \frac{\pi u_1(e) + (1-\pi)u_1(0)}{1-\delta} = \kappa^*,$$

contradicting our assumption that $\kappa^{**} > \kappa^*$. If $G^{**} = B^{**} < \kappa^{**}, b^{**} + e = g^{**}$ from P^2, and R^H holds. If $G^{**} > B^{**}$, we will show that R^H is slack. At $g' = b^{**} + e, G' = B^{**}$,

$$u_2(1+e-g') + \delta W(G') = U^2(c^*) = u_2(1+e-g^{**}) + \delta W(G^{**}).$$

Let g'' be defined by

$$u_1(g'') + \delta G' = u_1(g^{**}) + \delta G^{**};$$

$g'' \leq g'$ would imply

$$u_2(1+e-g'') + \delta W(G') \geq u_2(1+e-g') + \delta W(G') = U^2(c^*),$$

which, as $G' < G^{**}$, contradicts either uniqueness or optimality of the proposed solution g^{**}, G^{**}. Therefore, $g'' > g'$, and

$$u_1(g^{**}) + \delta G^{**} = u_1(g'') + \delta G' > u_1(g') + \delta G' = u_1(b^{**} + e) + \delta B^{**}.$$

$\square$

Now define

$$\begin{aligned} \gamma^R &: = u_1(b(\kappa^{**}) + e) + \delta B(\kappa^{**}), \\ \kappa^R &: = \pi\gamma^R + (1-\pi)\beta(\kappa^{**}). \end{aligned}$$

From step 1, if $G(\kappa^{**}) = B(\kappa^{**})$, $\gamma^R = u_1(g(\kappa^{**})) + \delta G(\kappa^{**})$ and therefore $\kappa^R = \kappa^{**}$; if $G(\kappa^{**}) > B(\kappa^{**})$, $\gamma^R < u_1(g(\kappa^{**})) + \delta G(\kappa^{**})$, and $\kappa^R < \kappa^{**}$.

Step 2: For all $\kappa \geq \kappa^R$, $(b(\kappa), B(\kappa)) = (b(\kappa^{**}), B(\kappa^{**}))$.

Proof of Step 2: If $\kappa^R = \kappa^{**}$, the claim is true by definition; we therefore proceed directly to the case $G(\kappa^{**}) > B(\kappa^{**})$, $\kappa^R < \kappa^{**}$. As $(b(\kappa^{**}), B(\kappa^{**}))$ is the (unique) solution to $V(\beta(\kappa^{**}); u_1, 1)$, P_L^2 is violated at any $(b, B) \neq (b(\kappa^{**}), B(\kappa^{**}))$ if $u_1(b)+\delta B = \beta(\kappa^{**})$. Similarly, P_L^2 must be violated at any $\beta(\kappa) > \beta(\kappa^{**})$. The claim above is therefore equivalent to $\beta(\kappa) = \beta(\kappa^{**})$ for $\kappa \geq \kappa^R$. Suppose $\beta(\kappa) < \beta(\kappa^{**})$ at $\kappa \geq \kappa^R$. Then

$$\gamma(\kappa) > \frac{1}{\pi}[\kappa - (1-\pi)\beta(\kappa^{**})] > u_1(g(\kappa^{**})) + \delta G(\kappa^{**}).$$

Given $\gamma(\kappa)$, if R^H holds at $b' = b(\beta(\kappa); u_1, 1)$, $B' = B(\beta(\kappa); u_1, 1)$, optimality of c^κ requires $(b', B') = (b(\kappa), B(\kappa))$. As $\beta(\kappa) < \beta(\kappa^{**})$ and $b(\kappa^{**}), B(\kappa^{**})$ solve $V(\beta(\kappa^{**}); u_1, 1)$, Lemma 5 implies that

$$u_1(b'+e) + \delta B' < u_1(b(\kappa^{**})+e) + \delta B(\kappa^{**}) < u_1(g(\kappa^{**})) + \delta G(\kappa^{**}) < \gamma(\kappa).$$

Hence, $(b', B') = (b(\kappa), B(\kappa))$, and R^H is slack at κ. Lemma 5 also implies that

$$u_2(1 - b(\kappa)) + \delta W(B(\kappa)) > u_2(1 - b(\kappa^{**})) + \delta W(B(\kappa^{**})) = U^2(c^*);$$

but by Lemma 4(ii) P_L^2 and R^H cannot both be slack, so that indeed $\beta(\kappa) = \beta(\kappa^{**})$, and P_L^2 binds at $\kappa \in [\kappa^R, \kappa^{**}]$.□

As $\gamma(\kappa) = \frac{1}{\pi}[\kappa - (1-\pi)\beta(\kappa^{**})]$ for $\kappa > \kappa^R$, Lemma 5 also implies that R^H is slack at all $\kappa > \kappa^R$. We now show that for $\kappa < \kappa^R$, R^H must be tight (it binds at κ^R by definition). Suppose R^H is slack at $\kappa < \kappa^R$. Then $b(\kappa), B(\kappa)$ solve $V(\beta(\kappa); u_1, 1)$ by Claim 2, as do $b(\kappa^R), B(\kappa^R)$ for $\beta(\kappa^R)$ by steps 1 and 2. P_L^2 again requires $\beta(\kappa) \leq \beta(\kappa^R)$. By uniqueness of the solution to V, $\beta(\kappa) = \beta(\kappa^R)$ implies $b(\kappa^R) = b(\kappa), B(\kappa^R) = B(\kappa)$. But then, as $\kappa = \pi\gamma(\kappa) + (1-\kappa)\beta(\kappa)$ and $\kappa < \kappa^R$, we must have $\gamma(\kappa) < \gamma(\kappa^R)$, which violates R^H. Therefore $\beta(\kappa) < \beta(\kappa^R)$, and as P_L^2 is tight at κ^R, Lemma 5 implies that P_L^2 is slack at κ. But by Lemma 4(ii) P_L^2 and R^H cannot both be slack. ■

Lemma 7 *P_H^2 is slack at all $\kappa < \kappa^{**}$.*

Proof. By Lemma 5 and Claim 1, P_H^2 is slack at all κ such that $\gamma(\kappa) < \gamma(\kappa^{**})$. We have already shown this to be true for $\kappa \geq \kappa^R$ in Step 2 of Lemma 6; we proceed with $\kappa < \kappa^R$.

We now show that in fact $\gamma(\kappa) \leq \gamma(\kappa^R)$ for $\kappa < \kappa^R$. Again, suppose the opposite. For $b' = b(\beta(\kappa); u_1, 1)$, $B' = B(\beta(\kappa); u_1, 1)$, Lemma 5 and

$$\beta(\kappa) = \frac{1}{1-\pi}[\kappa - \pi\gamma(\kappa)] < \frac{1}{1-\pi}[\kappa^R - \pi\gamma(\kappa^R)] = \beta(\kappa^{**})$$

imply

$$u_1(b' + e) + \delta B' < u_1(b(\kappa^{**}) + e) + \delta B(\kappa^{**}) = \gamma(\kappa^R) < \gamma(\kappa)$$

so that $(b', B') = (b(\kappa), B(\kappa))$, and

$$u_2(1 - b') + \delta W(B') > u_2(1 - b(\kappa^{**})) + \delta W(B(\kappa^{**})) = U^2(c^*),$$

which contradicts Lemma 4(ii). ■

Definition 6 *$z : \mathbb{R} \to \mathbb{R}$ is defined by $z(b) = \pi u_1(b + e) + (1 - \pi)u_1(b)$.*

$h : \mathbb{R} \to \mathbb{R}$ is defined by $h(b) = u_1(b + e) - u_1(b)$.

Definition 7 *$V_H : [\gamma(\kappa^*), \gamma(\kappa^{**})] \to \mathbb{R}$ is defined by $V_H(\gamma) = V(\gamma; u_1, 1 + e)$.*

$V_L : [0, 1] \times \mathcal{K} \to \mathbb{R}$ is defined by $V_L(b; \kappa) = u_2(1 - b) + \delta W(\frac{\kappa - z(b)}{\delta})$.

Definition 8 $b^P(\kappa) = \max b : u_2(1 - b) + \delta W(\frac{\kappa - z(b)}{\delta}) = U^2(c^*)$.

Lemma 8 *For all $\kappa \in [\kappa^*, \kappa^R]$, the problem $W(\kappa)$ can be written as*

$$\begin{aligned} W(\kappa) &= \max_b V(b; \kappa) := \pi V_H(\kappa + (1 - \pi)h(b)) + (1 - \pi)V_L(b; \kappa) \\ &\text{s.t. } b \leq \min\{z^{-1}(\kappa - \delta\kappa^*), 1, b^P(\kappa)\}. \end{aligned}$$

Proof. It is easy to see that wherever R^H binds, we can rewrite R^H and the adding up constraint as $\gamma = \kappa + (1 - \pi)h(b)$, $\kappa = z(b) + \delta B$. Agent 2's utility in the high state can be rewritten as $V_H(\gamma)$ by claim 1.

Step 1: $b(\kappa) > 0$ for all $\kappa > \kappa^*$.

Proof of Step 1: It is easy to see that the adding up constraint at κ^* together with R^H and F' determine $b(\kappa^*) = 0, B(\kappa^*) = \kappa^*$. For $\kappa \geq \kappa^R$, $b(\kappa) = b(\kappa^{**}) > 0$ as shown in Step 1 of Lemma 6. At $\kappa < \kappa^R$, R^H binds. Suppose there exists $\kappa > \kappa^*$ such that $b(\kappa) = 0$. Then define

$$\kappa^0 = \max \kappa \in \mathcal{B} : b(\kappa) = 0;$$

this exists as $b^* = 0$, $b(\kappa^{**}) > 0$ by Lemma 6, and b as a solution to $W(\kappa)$ is continuous by strict concavity of u_i, W. This also implies $\kappa^0 < \kappa^P$, so that R^H binds at κ^0. Then at $c^{\kappa^*}, c^{\kappa^0}$, we have

$$\kappa^* = z(0) + \delta B^*, \kappa^0 = z(0) + \delta B(\kappa^0).$$

Note that $\kappa^0 - \kappa^* > 0$ by assumption; therefore the above also implies $B(\kappa^0) - B^* > 0$. Then

$$\kappa^0 - \kappa^* = \delta(B(\kappa^0) - B^*) < B(\kappa^0) - B^* = B(\kappa^0) - \kappa^*,$$

as $\delta < 1$. Therefore, by definition of κ^0, $b(B(\kappa^0)) > 0 = b(\kappa^0)$. But we can show that efficiency requires $b(\kappa) \geq b(B(\kappa))$ for all κ. Suppose not. Then a change $b(\kappa) + \Delta, b(B(\kappa)) - \Delta^B)$ such that

$$u_1(b(\kappa)) + \delta(1-\pi)u_1(b(B(\kappa)) = u_1(b(\kappa) + \Delta) + \delta(1-\pi)u_1(b(B(\kappa) - \Delta^B)$$

still satisfies P^2 and R^H at $B(\kappa)$ for small enough Δ and constitutes a Pareto improvement. To show that it satisfies R^H at β, note that by strict concavity of u_1,

$$\begin{aligned} u_1(b(\kappa) + e) - u_1(b(\kappa) + \Delta + e) &< u_1(b(\kappa)) - u_1(b(\kappa) + \Delta) = \\ &= \delta(1-\pi)[u_1(b(B(\kappa)) - u_1(b(B(\kappa) - \Delta^B)] \end{aligned}$$

which implies

$$\begin{aligned} \gamma(\kappa) &\geq u_1(b(\kappa) + e) + \delta(1-\pi)u_1(b(B(\kappa)) > \\ &> u_1(b(\kappa) + \Delta + e) + \delta(1-\pi)u_1(b(B(\kappa) - \Delta^B) \end{aligned}$$

Therefore $\kappa^0 = \kappa^*$.□

Step 2: For all κ, $B(\kappa) \leq B(\kappa^{**}) < \kappa^{**}$.

Proof of Step 2: In Lemma 6, we showed that $B(\kappa^R) = B(\kappa^{**}) < \kappa^{**}$. Now suppose there exists $\kappa < \kappa^R$ such that $B(\kappa) > B(\kappa^{**})$. As R^H binds at κ, κ^R, $\kappa = z(b(\kappa)) + \delta B\left(\kappa\right)$, $\kappa^R = z(b(\kappa^R)) + \delta B\left(\kappa^R\right)$, implying $b(\kappa) < b(\kappa^R)$. But note that $(b(\kappa^R), B(\kappa^R))$ solve $V(\beta(\kappa^{**}); u_1, 1)$, and that P_L^2 therefore requires $\beta(\kappa) < \beta^R$. Letting (b', B') denote the solution to $V(\beta(\kappa); u_1, 1)$, by Lemma 5 we have $B(\kappa) > B(\kappa^R) \geq B'$. As

$$u_1(b(\kappa)) + \delta B(\kappa) = \beta(\kappa) = u_1(b') + \delta B',$$

this implies $b(\kappa) < b'$, contradicting Claim 2.□

Step 3: If P_L^2 binds at κ, $b(\kappa) = b^P(\kappa)$.

Proof of Step 3: Due to strict concavity of z, u_2, W, for any κ there exist at most two points $b_P(\kappa), b^P(\kappa)$ where $u_2(1-b) + \delta W(\frac{\kappa - z(b)}{\delta}) = U^2(c^*)$, $b \in \{b_P(\kappa), b^P(\kappa)\}$ with $b_P(\kappa) \leq b^P(\kappa)$. Again due to strict concavity, it is easy to see that $b_P(\kappa)$ is strictly increasing,$b^P(\kappa)$ strictly decreasing in κ. By definition of κ^R, we have

$$u_2(1 - b(\kappa^R)) + \delta W(\frac{\kappa - z(b(\kappa^R))}{\delta}) = U^2(c^*).$$

Therefore, if $b(\kappa^R) = b_P(\kappa)$, $b_P(\kappa) < b(\kappa^R)$, for all $\kappa < \kappa^R$; whereas if $b(\kappa^R) = b^P(\kappa)$, $b_P(\kappa) < b_P(\kappa^R) \leq b(\kappa^R)$, for all $\kappa < \kappa^R$. Therefore, as

$$u_2(1 - b(\kappa^R)) + \delta W(\frac{\kappa - z(b(\kappa^R))}{\delta}) = u_2(1 - b_P(\kappa)) + \delta W(\frac{\kappa - z(b_P(\kappa))}{\delta}),$$

we must have

$$B(\kappa^R) = \frac{\kappa - z(b(\kappa^R))}{\delta} < \frac{\kappa - z(b_P(\kappa))}{\delta},$$

implying that if $b(\kappa) = b_P(\kappa)$ at some $\kappa < \kappa^R$, we would also have $B(\kappa) > B(\kappa^R)$, contradicting Step 2.□

As z is continuous and monotone, the inverse z^{-1} is well defined, and given the definition of V_H, the constraint set of W indeed reduces to $b \leq \min\{z^{-1}(\kappa - \delta\kappa^*), 1, b^P(\kappa)\}$.
■

It is easy to see that $V^H(\gamma)$ is strictly concave by strict concavity of u_i, W; therefore the subdifferential $\partial V^H(\gamma)$ exists for all $\gamma \in [\gamma(\kappa^*), \gamma(\kappa^{**})]$. By Theorem 10.6 of Rockafellar and Wets (1998), the subdifferential $\partial V(b; \kappa)$ exists by strict concavity of W and differentiability of u_i and can be written as

$$\begin{aligned} \partial V(b;\kappa) = \{v | v =& (1-\pi)[\pi v_H h'(b) - u_2'(1-b) - v_B z'(b)], \\ & v_H \in \partial V^H(\kappa + (1-\pi)h(b)), v_B \in \partial W(\frac{\kappa - z(b)}{\delta})\}. \end{aligned}$$

Optimality of $b(\kappa)$ then requires that there exists

$$v_e \in \partial V(b(\kappa);\kappa) : v_e \geq 0; \text{ with equality if } b(\kappa) < \min\{z^{-1}(\kappa - \delta\kappa^*), 1, b^P(\kappa)\}.$$

(In the following, v_e will always denote the specific element of $\partial V(b;\kappa)$ that satisfies these conditions).

Lemma 9 *For $y > y'$ and $x > 0$, $y, y', x \in \mathbb{R}$, agents' marginal utilities satisfy*

$$\frac{u_i'(y)}{u_i'(y+x)} \leq \frac{u_i'(y')}{u_i'(y'+x)}.$$

Proof. Define, for a simple lottery p with prizes y, y', its certainty equivalent $C(p)$ according to utility function u_i as

$$u_i(C(p)) = pu_i(y) + (1-p)u_i(y').$$

Slightly abusing notation, $p+x$ will denote the simple lottery p with prizes $y+x, y'+x$. It is easy to see that nonincreasing absolute risk aversion implies

$$C(p) \leq C(p+x) - x.$$

Define $u_i^x : \mathbb{R} \to \mathbb{R}$ by

$$u_i^x(c) = u_i(c+x).$$

As u_i, u_i^x are ordinally equivalent, there always exists some increasing function ξ such that $u_i = \xi \circ u_i^x$; we now show that $C(p) + x \leq C(p+x)$ implies concavity of ξ. For any $p \in (0,1)$, and any two points $u_i(y+x)$, $u_i(y'+x)$,

$$\begin{aligned}
\xi(pu_i^x(x) + (1-p)u_i^x(y)) &= \xi(u_i(C(p+x)) = \xi(u_i^x(C(p+x)-x)) = \\
&= u_i(C(p+x)-x) \geq u_i(C(p)) = pu_i(y) + (1-p)u_i(y') = \\
&= p\xi(u_i^x(y)) + (1-p)\xi(u_i^x(y')).
\end{aligned}$$

Differentiability of u_1 implies that ξ must be differentiable as well, and we therefore have

$$\frac{u_1'(y)}{u_1'(y+x)} = \xi'(u_1(y+x)) \leq \xi'(u_1(y'+x)) = \frac{u_1'(y')}{u_1'(y'+x)}$$

whenever $y > y'$ by concavity of ξ. ■

Lemma 10 *There exists* κ^P, $\kappa^* < \kappa^P \leq \kappa^R$ *such that* P_L^2 *binds at all* $\kappa \geq \kappa^P$ *and is slack elsewhere.*

Proof. Step 1: If $\kappa < \kappa', b(\kappa) \geq b(\kappa')$ and R^H binds, $\gamma(\kappa) < \gamma(\kappa')$.

Proof of Step 1: As

$$\kappa = z(b(\kappa)) + \delta B(\kappa), \kappa' = z(b(\kappa')) + \delta B(\kappa')$$

by $R^H, b(\kappa) \geq b(\kappa')$ implies $B(\kappa) < B(\kappa')$. By strict concavity of u_1, if $\gamma(\kappa) \geq \gamma(\kappa')$,

$$u_1(b(\kappa)) - u_1(b(\kappa')) \geq u_1(b(\kappa)+e) - u_1(b(\kappa')+e) \geq \delta(B(\kappa') - B(\kappa))$$

or $\beta(\kappa) \geq \beta(\kappa')$, which contradicts $\kappa < \kappa'$.□

Step 2: If $\kappa, \kappa' \in (\kappa^*, \kappa^R)$ with $\kappa < \kappa'$, and $b(\kappa) \geq b(\kappa')$, it must be true that

$$\frac{1}{h'(b(\kappa))} v_e > \frac{1}{h'(b(\kappa'))} v'_e, \text{for } v_e \in \partial V(b(\kappa); \kappa), v'_e \in \partial V(b(\kappa'); \kappa').$$

Proof of Step 2: By Step 1 and as $\kappa < \kappa'$, $b(\kappa) \geq b(\kappa')$, and $\kappa, \kappa' \in (\kappa^*, \kappa^R)$, we must have $\gamma(\kappa) < \gamma(\kappa')$. By strict concavity of V^H, this implies $v_H > v'_H$, for all $v_H \in \partial V^H(\gamma(\kappa))$, $v'_H \in \partial V^H(\gamma(\kappa'))$. $\kappa, \kappa' \in (\kappa^*, \kappa^R)$ and $\kappa < \kappa', b(\kappa) \geq b(\kappa')$ imply $B(\kappa) < B(\kappa')$. If $b(\kappa) = b(\kappa')$,

$$\frac{1}{h'(b(\kappa))} v_e - \frac{1}{h'(b(\kappa'))} v'_e = \frac{z'(b(\kappa))}{-h'(b(\kappa))} [v_{B(\kappa)} - v_{B(\kappa')}] > 0,$$

for all $v_{B(\kappa)} \in \partial W(B(\kappa))$, $v_{B(\kappa')} \in \partial W(B(\kappa'))$, as $B(\kappa) < B(\kappa')$. We therefore proceed with the case $b(\kappa) > b(\kappa')$, which together with $B(\kappa) < B(\kappa')$ implies

$$\frac{u'_2(1 - b(\kappa))}{z'(b(\kappa))} + v_{B(\kappa)} > \frac{u'_2(1 - b(\kappa'))}{z'(b(\kappa'))} + v_{B(\kappa')},$$

for all $v_{B(\kappa)} \in \partial W(B(\kappa))$, $v_{B(\kappa')} \in \partial W(B(\kappa'))$.

Case 1: $b(\kappa') < b^P(\kappa')$. As $B(\kappa) < B(\kappa')$, and $b(\kappa') < b(\kappa) \leq 1$ by assumption, the first order condition holds with equality at κ' :

$$\frac{u'_2(1 - b(\kappa'))}{z'(b(\kappa'))} = -v^e_{B(\kappa')} + \pi h'(b(\kappa')) v^e_H \geq -v^e_{B(\kappa')}$$

(as $h' < 0$ by strict concavity of u_1). Therefore,

$$\frac{u'_2(1 - b(\kappa))}{z'(b(\kappa))} + v^e_{B(\kappa)} > \frac{u'_2(1 - b(\kappa'))}{z'(b(\kappa'))} + v^e_{B(\kappa')} \geq 0.$$

Case 2: $b(\kappa') = b^P(\kappa')$. $\kappa' < \kappa^R$ and Step 2 of Lemma 8 then imply $B(\kappa) < B(\kappa^R) < \kappa^{**}$, $b(\kappa) > b(\kappa^R) > 0$. From the characterization of $b(\kappa^R)$, $B(\kappa^R)$, we know that there exists $v_{B(\kappa^R)} \in \partial W(B(\kappa^R))$ such that

$$\frac{u'_2(1 - b(\kappa^R))}{u'_1(b(\kappa^R))} = -v_{B(\kappa^R)}.$$

Therefore,

$$\frac{u'_2(1 - b(\kappa'))}{z'(b(\kappa'))} > \frac{u'_2(1 - b(\kappa'))}{u'_1(b(\kappa'))} > \frac{u'_2(1 - b(\kappa^R))}{u'_1(b(\kappa^R))} = -v_{B(\kappa^R)} > -v_{B(\kappa')},$$

for all $v_{B(\kappa')} \in \partial W(B(\kappa'))$, which again implies

$$\frac{u'_2(1 - b(\kappa))}{z'(b(\kappa))} + v^e_{B(\kappa)} > \frac{u'_2(1 - b(\kappa'))}{z'(b(\kappa'))} + v^e_{B(\kappa')} \geq 0.$$

It is easy to see that

$$\frac{z'(b(\kappa))}{-h'(b(\kappa))} \geq \frac{z'(b(\kappa'))}{-h'(b(\kappa'))} \text{ if and only if } \frac{u_1'(b(\kappa)+e)}{u_1'(b(\kappa')+e)} \geq \frac{u_1'(b(\kappa))}{u_1'(b(\kappa'))},$$

which follows from Lemma 9 and $b(\kappa) > b(\kappa')$. Therefore,

$$\frac{z'(b(\kappa))}{-h'(b(\kappa))}[\frac{u_2'(1-b(\kappa))}{z'(b(\kappa))} + v^e_{B(\kappa)}] - \frac{z'(b(\kappa'))}{-h'(b(\kappa'))}[\frac{u_2'(1-b(\kappa'))}{z'(b(\kappa'))} + v^e_{B(\kappa')}] > 0.$$

□

Step 3: If there exists $\kappa < \kappa^R$ such that $b(\kappa) = b^P(\kappa)$, then $b(\kappa') = b^P(\kappa')$ for all $\kappa' \in [\kappa, \kappa^R]$.

Proof of Step 3: Suppose not, for some $\kappa \in (\kappa, \kappa^R)$. Then $b(\kappa') < b^P(\kappa') < b^P(\kappa) = b(\kappa) \leq 1$ as $\kappa < \kappa'$, and therefore also $B(\kappa') > B(\kappa) \geq \kappa^*$. Then optimality requires that there exist $v_e \in \partial V(b(\kappa);\kappa), v_e' \in \partial V(b(\kappa');\kappa')$ such that

$$v_e \geq 0, v_e' = 0; \text{ implying } \frac{1}{h'(b(\kappa'))}v_e' = 0 \geq \frac{1}{h'(b(\kappa))}v_e,$$

contradicting Step 2. □

Case 1: $b(\kappa^R) = 1$ or $B(\kappa^R) = \kappa^*$. Define $\kappa^P := \kappa^R$. For all $\kappa < \kappa^P$, $B(\kappa) \leq B(\kappa^P)$ by Step 2 of Lemma 8, implying $B(\kappa) = \kappa^*$ for all κ if $B(\kappa^R) = \kappa^*$. Hence, $b(\kappa) \leq b(\kappa^P)$ necessarily and with one inequality strict due to the adding up constraint $\kappa = z(b) + \delta B$. Therefore P_L^2 is slack for all $\kappa < \kappa^P$ and binds above by Step 2 of Lemma 6.

Case 2: $b(\kappa^R) < 1$ and $B(\kappa^R) > \kappa^*$, but there exists κ with $b^P(\kappa) = 1$ and $v > 0$ for all $v \in \partial V(b^P(\kappa);\kappa)$.

Define κ^P by $b^P(\kappa^P) = 1$. P_L^2 binds at all $\kappa \geq \kappa^P$ by Step 3, and Step 2 of Lemma 6. $\kappa^P < \kappa^R$ as b^P is strictly decreasing. If P_L^2 were to bind at any $\kappa < \kappa^P$, we must have $b(\kappa) = b_P(\kappa)$ as $b^P(\kappa^P) = 1$ and b^P is strictly decreasing. But this has been ruled out in Step 3 of Lemma 8.

Case 3: $b(\kappa^R) < 1$ and $B(\kappa^R) > \kappa^*$, but there exists κ with $b^P(\kappa) = z^{-1}(\kappa - \delta\kappa^*) < 1$ and $v > 0$ for all $v \in \partial V(b^P(\kappa);\kappa)$.

Define κ^P by $b^P(\kappa) = z^{-1}(\kappa - \delta\kappa^*)$. P_L^2 binds at all $\kappa \geq \kappa^P$ by Step 3. $\kappa^P < \kappa^R$ as $B(\kappa^P) = \kappa^* < B(\kappa^R)$. If P_L^2 were to bind at any $\kappa < \kappa^P$, we must have $b(\kappa) = b_P(\kappa)$ as b^P is strictly decreasing, again contradicting Step 3 of Lemma 8.

Case 4: $b(\kappa^R) < 1$ and $B(\kappa^R) > \kappa^*$, but there exists κ such that $v_e = 0$, for $v_e \in \partial V(b^P(\kappa);\kappa)$.

Note that claim 2 implies that $v \geq 0$, $v \in \partial V(b(\kappa);\kappa)$ for any κ such that $b(\kappa) = b(\beta(\kappa); u^1, 1)$. Therefore there exists $v \in \partial V(b(\kappa^R);\kappa^R)$ such that $v \geq 0$. Using step 3, it is easy to see that $v > v'$, for all $v \in \partial V(b^P(\kappa);\kappa)$, $v' \in \partial V(b^P(\kappa');\kappa')$. These 4 cases therefore exhaust all possibilities. Define

$$\kappa^P = \kappa \in [\kappa^*, \kappa^R] \text{ such that there exists } v_e \in \partial V(b^P(\kappa);\kappa) \text{ with } v_e = 0.$$

$\kappa^P \leq \kappa^R$ by the above and the definition of κ^P. Necessarily, $b(\kappa^P) = b^P(\kappa^P)$, and P_H^2 binds for all $\kappa \in [\kappa^P, \kappa^R]$ by Step 3. We can then again apply Step 2 to show that there can not exist $\kappa < \kappa^P$ such that $b(\kappa) = b^P(\kappa)$.

In all three cases, as $b(\kappa^P) \geq b(\kappa^R) > 0$ by Lemma 6 and $b(\kappa^*) = 0$, we have $\kappa^P > \kappa^*$. ■

Lemma 11 *If $b(\kappa) = 1$ at any κ, there exists κ^1, $\kappa^* < \kappa^1 \leq \kappa^P$ such that $b(\kappa) = 1$ for all $\kappa \in [\kappa^1, \kappa^P]$, and $b(\kappa) < 1$ everywhere else.*

Proof. We first show that if $b(\kappa) = 1$ at any $\kappa < \kappa^P$, then $b(\kappa') = 1$ for all $\kappa' \in (\kappa, \kappa^P)$. Suppose not. Then $b(\kappa') < 1 = b(\kappa)$, and $B(\kappa') > B(\kappa) \geq \kappa^*$ as $\kappa' > \kappa$. We can again use Step 2 of Lemma 10 to derive a contradiction.

Case 1: $b(\kappa^R) = 1$, $B(\kappa^R) = \kappa^*$. In this case, $\kappa^R = \kappa^P$, $B(\kappa) = \kappa^*$ for all κ, and $b(\kappa) < 1$ for all $\kappa < \kappa^R$ as $\kappa = z(b(\kappa)) + \delta B(\kappa)$.

Case 2: For $\kappa = z(1) + \delta\kappa^*$, $v > 0$ for all $v \in \partial V(1;\kappa)$.

Define $\kappa^1 = z(1) + \delta\kappa^*$. For any $\kappa < \kappa^1$, $B(\kappa) \geq \kappa^*$ implies $b(\kappa) < 1$. By the above, $b(\kappa) = 1$ for all $\kappa \in [\kappa^1, \kappa^P]$.

Case 3: Define $\kappa^1 := \kappa \in [\kappa^*, \kappa^P]$ such that there exists $v_e \in \partial V(1;\kappa)$ with $v_e = 0$. By the above, $b(\kappa) = 1$ for all $\kappa \in [\kappa^1, \kappa^P]$. $b(\kappa) < 1$ for $\kappa < \kappa^1$ can again be derived by contradiction from Step 2 of Lemma 10.

In either case, $\kappa^1 > \kappa^*$ follows from $b(\kappa^*) = 0$. ■

Definition 9 *If $b(\kappa) < 1$ for all κ, define $\kappa^1 := \kappa^P$.*

Lemma 12 *(i) For $\kappa \in int\mathcal{K}$, $g(\kappa) \in (0, 1+e)$; (ii) for $\kappa \in int\mathcal{K}$, $G(\kappa) \in int\mathcal{K}$.*

Proof. (i) Step 1: For all $\kappa, \kappa' \in \mathcal{K}$, $\gamma(\kappa) \leq \gamma(\kappa')$ whenever $\kappa < \kappa'$.

Proof of Step 1: In Lemma 6 we showed γ to be strictly increasing on $[\kappa^R, \kappa^{**}]$. The proof of Lemma 7 showed that $\gamma(\kappa) \leq \gamma(\kappa^R)$, for all $\kappa < \kappa^R$. If $\kappa^P < \kappa^R$, it is easy to see from decreasingness of b^P and h that $\gamma(\kappa) = \kappa + (1-\pi)h(b^P(\kappa))$ is strictly increasing on $[\kappa^P, \kappa^R]$. If $\kappa^P > \kappa^1$, increasingness of γ on $[\kappa^1, \kappa^P]$ follows from $\kappa + (1-\pi)h(1)$. As $b^P(\kappa^P) = 1$ in this case, we also have $\gamma(\kappa) < \gamma(\kappa^P)$ in this range. For $\kappa < \kappa'$ such that $\kappa' \leq \kappa^1$, suppose $\gamma(\kappa) > \gamma(\kappa')$. By Step 1 of Lemma 10, this implies $b(\kappa) < b(\kappa')$ and therefore also $B(\kappa) > B(\kappa') \geq \kappa^*$. We again derive a contradiction from optimality and Step 2 of Lemma 10. □

Step 2: $g(\kappa^{**}) = e$.

Proof of Step 2: P^2 implies $g(\kappa^{**}) = e$ if $G(\kappa^{**}) = \kappa^{**}$. Suppose $G(\kappa^{**}) < \kappa^{**}$. Then Lemma 7 implies that P^2_H is slack at $G(\kappa^{**})$. If $g(\kappa^{**}) > g(G(\kappa^{**}))$, a change $g(\kappa^{**}) - \Delta, g(G(\kappa^{**})) + k\Delta$ with

$$\Delta \in (\frac{u_1'(g(\kappa^{**}))}{\delta\pi u_1'(g(G(\kappa^{**})))}, \frac{u_2'(1+e-g(\kappa^{**}))}{\delta\pi u_2'(1+e-g(G(\kappa^{**})))})$$

still satisfies P^2 and R^H both at κ^{**} and at $G(\kappa^{**})$ for small enough Δ and constitutes a Pareto improvement. Therefore, $g(\kappa^{**}) \leq g(G(\kappa^{**}))$. Then Lemma 5, $g(\kappa^{**}) > e > 0$ (by P^2) and $\gamma(\kappa^{**}) > \gamma(G(\kappa^{**}))$ imply $g(\kappa^{**}) = g(G(\kappa^{**})) = 1 + e$. We note that if $\kappa^{**} > G(\kappa^{**}) > B(\kappa^{**}), g(\kappa^{**}) < b(\kappa^{**}) + e \leq 1 + e$ by P^2, which is a contradiction; therefore if $G(\kappa^{**}) > B(\kappa^{**}), G(\kappa^{**}) = \kappa^{**}$. If $\kappa^{**} > G(\kappa^{**}) = B(\kappa^{**}), g(\kappa^{**}) = b(\kappa^{**}) + e$ by P^2, which by the above reasoning implies $g(\kappa^{**}) = 1 + e, b(\kappa^{**}) = 1$. If $G(\kappa^{**}) = B(\kappa^{**}), \kappa^R = \kappa^{**}$, and therefore by Lemma 6, R^H binds for all κ, while P^2_H is slack for all $\kappa < \kappa^{**}$, which by Lemma 4(i) implies $G(G(\kappa^{**})) \geq B(G(\kappa^{**}))$.If R^H binds at some κ, $G(\kappa) \geq B(\kappa)$ implies $g(\kappa) \leq b(\kappa) + e$. But then $g(\kappa) = 1 + e$ implies $b(\kappa) = 1$ and $G(\kappa) = B(\kappa)$ for all κ. As $g(\kappa) \leq g(G(\kappa))$, we have $g(G(\kappa)) = 1 + e$, which again implies $b(G(\kappa)) = 1$ and $G(G(\kappa)) = B(G(\kappa))$; and as $G(\kappa) = B(\kappa)$, uniqueness of c^κ then also implies $g(B(\kappa)) = 1 + e$, $b(B(\kappa)) = 1$ and $G(B(\kappa)) = B(B(\kappa))$. Therefore, $\kappa^{**} > G(\kappa^{**}) = B(\kappa^{**})$, $g(\kappa^{**}) = 1 + e, b(\kappa^{**}) = 1$ imply by induction on the event tree that

$$W(\kappa^{**}) = \frac{u_2(0)}{1-\delta} < \frac{u_2(1)}{1-\delta} U^2(c^*),$$

which is a contradiction. Therefore $\kappa^{**} = G(\kappa^{**}) > B(\kappa^{**})$, and $g(\kappa^{**}) = e$. □

Step 3: $g(\kappa) > 0$ for all $\kappa > \kappa^*$.

Proof of Step 3: We have shown $b(\kappa) > 0$ for all $\kappa > \kappa^*$ in Step 1 of Lemma 8. As P^2_H is slack at $\kappa \in int\mathcal{K}$, Lemma 4(i) then implies that $g(\kappa) > 0, \kappa > \kappa^*$. □

That $g(\kappa) \in (0, e)$ for all $\kappa \in int\mathcal{K}$ now follows from Claim 1, Lemma 5 and Step 1-3.

(ii) Step 4: $G(\kappa^*) > \kappa^*$.

Proof of Step 4: We see from R^H with $b(\kappa^*) = 0$, $B(\kappa^*) = \kappa^*$ that

$$u_1(g(\kappa^*)) + \delta G(\kappa^*) = u_1(e) + \delta\kappa^*;$$

so that $G(\kappa^*) = \kappa^*$, implies $g(\kappa^*) = e$ and therefore $W(\kappa^*) = W(\kappa^{**})$, contradicting $\kappa^* < \kappa^{**}$ or strict decreasingness of W. □

Step 5: For $\kappa \in [\kappa^R, \kappa^{**}]$, $G(\kappa) = \kappa$.

Proof of Step 5: First note that we have shown $\kappa^R < \kappa^{**}$ in Step 2. The proof of $g(\kappa^{**}) \leq g(G(\kappa^{**}))$ used in Step 2 can be applied to all $\kappa \in (\kappa^R, \kappa^{**})$, implying $g(\kappa) \leq g(G(\kappa))$, and by monotonicity of γ and therefore g in this region, $\kappa \leq G(\kappa)$ for all κ. Suppose for $\kappa \in (\kappa^R, \kappa^{**}), \kappa < G(\kappa)$, Then R^H is slack at $G(\kappa)$, P_H^2 is slack at κ, and $g(\kappa) < g(G(\kappa))$, so that a change $g(\kappa) + \Delta, g(G(\kappa)) - k\Delta$ with

$$k \in \left(\frac{u_2'(1+e-g(\kappa))}{\delta\pi u_2'(1+e-g(G(\kappa)))}, \frac{u_1'(g(\kappa))}{\delta\pi u_1'(g(G(\kappa)))}\right)$$

still satisfies P^2 and R^H both at κ and at $G(\kappa)$ for small enough Δ and constitutes a Pareto improvement. Therefore $G(\kappa) = \kappa$ for $\kappa \in (\kappa^R, \kappa^{**})$. But as γ continuous and G is a continuous function of κ, we must also have $G(\kappa^R) = \kappa^R$. □

That $G(\kappa) \in int\mathcal{K}$ for all $\kappa \in int\mathcal{K}$ now follows from Claim 1, Lemma 5, Step 1,4 and 5, and $\kappa^R < \kappa^{**}$. ■

Lemma 13 *V_H is differentiable, and at all κ,*

$$V_H'(\gamma(\kappa)) = \frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))}.$$

Proof. Follows from Lemma 12, boundedness and concavity of u_i, W and Theorem 10.13 of Rockafellar and Wets (1998). ■

We can now show

$$\frac{u_1'(g(\kappa))}{u_1'(b(\kappa))} < \frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))}, \text{ for all } \kappa.$$

Case 1: $\kappa = \kappa^{**}$.

From Lemmas 12(i) and 6, $\kappa^{**} = G(\kappa^{**}) > B(\kappa^{**})$, and $g(\kappa^{**}) = e$. From the characterization of $b(\kappa^{**})$, $B(\kappa^{**})$ in Lemma 6 and the first order conditions in Lemma 5, we then have for $v_B \in \partial W(B(\kappa^{**}))$, $v_G \in \partial W(G(\kappa^{**}))$,

$$\frac{u_2'(1+e-g(\kappa^{**}))}{u_1'(g(\kappa^{**}))} \geq -v_G > -v_B \geq \frac{u_2'(1-b(\kappa^{**}))}{u_1'(b(\kappa^{**}))},$$

which yields the desired result.

Lemma 4(i) and Step 1 of Lemma 8 imply

$$\frac{u_1'(g(\kappa))}{u_1'(b(\kappa))} \leq \frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))}$$

for all $\kappa \in int\mathcal{K}$. We proceed to show that this inequality must be strict in an efficient sharing rule.

Case 2: $\kappa \in (\kappa^*, \kappa^P)$.

Suppose

$$\frac{u_1'(g(\kappa))}{u_1'(b(\kappa))} = \frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))}.$$

This implies $g(\kappa) < b(\kappa) + e$; to satisfy R^H, it is then necessary to have $G(\kappa) > B(\kappa)$. Define $\mathcal{A} = [0, 1+e] \times [0, 1] \times \mathcal{K}^2$, and let

$$\begin{aligned}
a(g, b, G, B) &= \pi[u_2(1+e-g) + \delta W(G)] + (1-\pi)[u_2(1-b) + \delta W(B)], \\
a_1(g, b, G, B) &= \pi[u_1(g) + \delta G] + (1-\pi)[u_1(b) + \delta B], \\
a_2(g, b, G, B) &= u_1(g) + \delta G - u_1(b+e) - \delta B, \\
a_3(g, G) &= u_2(1+e-g) + \delta W(G) \\
a_4(b, B) &= u_2(1-b) + \delta W(B)
\end{aligned}$$

Let P^κ denote the tuple $(g(\kappa), b(\kappa), G(\kappa), B(\kappa))$. Agent 2's level set at P^κ is then

$$A_2 = \{(g, b, G, B) \in \mathcal{A} | a(g, b, G, B) \geq W(\kappa)\};$$

we can also define the constraint set as

$$\begin{aligned}
A_1 = \{(g, b, G, B) \in \mathcal{A} | a_1(g, b, &G, B) \geq \kappa, a_2(g, b, G, B) \geq 0, \\
&a_3(g, G) \geq U^2(c^*), a_4(b, B) \geq U^2(c^*)\}.
\end{aligned}$$

We let $N_{A_i}(P^\kappa)$ denote the normal cone of the set A_i at the point P^κ. As A_2 is the level set of a concave function, we have

$$N_{A_2}(P^\kappa) = \{v \in \mathbb{R}^4 | v^T = -\partial a(P^\kappa)\},$$

Note that the subdifferential ∂W exists by strict concavity of W. The functions a_1, a_2 are of class C^1, and hence

$$N_{A_2}(P^\kappa) = \{v \in \mathbb{R}^4 | v = -\mu_1 \nabla a_1(P^\kappa) - \mu_2 \nabla a_2(P^\kappa) - \mu_3 \partial a_3(P^\kappa) - \mu_4 \partial a_4(P^\kappa) + z, z \in N_{\mathcal{C}}(P^\kappa)\}.$$

But for $\kappa < \kappa^P, P^2$ is slack by Lemmas 7 and 10, and therefore $\mu_3 = \mu_4 = 0$. The vector z with elements z^g, z^b, z^G, z^B is characterized by $z^b \geq 0, z^g \leq 0$ as $0 < b(\kappa), g(\kappa) < e$ by Lemmas 8 and 12(i), and $z^G \geq 0, z^B \leq 0$ as $\kappa^* \leq B(\kappa) < G(\kappa) \leq \kappa^{**}$. The tangent cones $T_{A_i}(P^\kappa)$ at the point c^κ are defined by

$$T_{A_i}(P^\kappa) = \{w | v \cdot w \leq 0, \text{ for all } v \in N_{A_i}(P^\kappa)\};$$

and $w \in int[T_{A_i}(P^\kappa)]$ if and only if the above inequality is strict. It is easy to verify that a vector w defined by

$$\begin{aligned} w &= \begin{bmatrix} w^g & w^b & w^G & w^B \end{bmatrix}, \text{ with} \\ w^b &= -\frac{\pi}{1-\pi}\frac{u_1'(g(\kappa))}{u_1'(b(\kappa))} w^g \text{ and} \\ w^B &= -\frac{\pi}{1-\pi} w^G \end{aligned}$$

satisfies $w \in T_{A_1}(P^\kappa)$ if

$$\delta w^G = -\frac{u_1'(g(\kappa))}{u_1'(b(\kappa))} z'(b(\kappa)) w^g.$$

Let this be true and take $w^g > 0$ and small; then for any $v \in N_{A_2}(P^\kappa)$,

$$\begin{aligned} & v \cdot w = \\ = \; & \pi w^g [\frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))} - \frac{u_1'(g(\kappa))}{u_1'(b(\kappa))}] u_2'(1-b(\kappa)) + \delta \pi w^G [v_B - v_G] = \\ = \; & \delta \pi w^G [v_B - v_G], \text{ with } v_G \in \partial W(G(\kappa)), v_B \in \partial W(B(\kappa)). \end{aligned}$$

Strict concavity of W and $G(\kappa) > B(\kappa)$ imply $v_G < v_B$, for all $v_G \in \partial W(G(\kappa)), v_B \in \partial W(B(\kappa))$. We have defined $w^G < 0$ above; we can therefore conclude $v \cdot w < 0$, for $w \in T_{A_1}(P^\kappa)$ and all $v \in N_{A_2}(P^\kappa)$. Then we can find $w' \in B_\varepsilon(w)$ such that $w' \in int[T_{A_i}(P^\kappa)], i = 1, 2$. As both sets are regular and locally closed, this implies by Theorem 6.36 of Rockafellar and Wets (1998) that $P^\kappa + w' \in A_i, i = 1, 2$, which contradicts either uniqueness or optimality of P^κ.

Case 3: $\kappa \in [\kappa^P, \kappa^{**})$.

For $\kappa \in [\kappa^P, \kappa^{**})$, it suffices to show that for the vector w defined above, we also have $v \cdot w \leq 0$ for all $v \in -\partial a_4(P^\kappa)$, and therefore $w \in T_{A_1}(P^\kappa)$ still. We showed in Lemma 12 that $(g(\kappa), G(\kappa)) \in (0, 1+e) \times int\mathcal{K}$ for $\kappa \in [\kappa^R, \kappa^{**})$. From the first order condition for $V(\gamma(\kappa); u_1, [0, 1+e] \times \mathcal{K})$, there then exists $v_G' \in \partial W(G(\kappa))$ such that

$$-v_G' = \frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))}.$$

By assumption, $G(\kappa) > B(\kappa)$, and therefore, for all $v \in -\partial a_4(P^\kappa), v_B \in \partial W(B(\kappa))$

$$\begin{aligned} v \cdot w &= u_2'(1-b(\kappa))w^b - \delta v_B w^B = \\ &= u_1'(b)\left[\frac{u_2'(1-b(\kappa))}{u_1'(b)} + \frac{z'(b(\kappa))}{u_1'(b)} v_B\right] w_B. \end{aligned}$$

But note that $G(\kappa) > B(\kappa)$,

$$\frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))} = \frac{u_1'(g(\kappa))}{u_1'(b(\kappa))}$$

by assumption and $w_B < 0$; therefore

$$\begin{aligned} v \cdot w &< u_1'(b)\left[\frac{u_2'(1-b(\kappa))}{u_1'(b)} + \frac{z'(b(\kappa))}{u_1'(b)} v_G'\right] w_B = \\ &= u_1'(b)\left[1 + \frac{z'(b(\kappa))}{u_1'(b)}\right] \frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))} w_B < 0. \end{aligned}$$

Case 4: $\kappa = \kappa^*$.

The proof that $m^1(g(\kappa^*), b(\kappa^*)) < m^2(g(\kappa^*), b(\kappa^*))$ requires results from Proposition 3 and will be delivered in Lemma 18. As we are not using this result in the proof of Proposition 3, there is no circularity.■

Lemma 14 *$\mathcal{E}(\mathcal{C}) = \mathcal{E}(\mathcal{C}_0) = \{c | c \in S(\kappa)$ for $\kappa \in \mathcal{K}\}$.*

Proof. $\mathcal{E}(\mathcal{C}_0) = \{c | c \in S(\kappa)$ for $\kappa \in \mathcal{K}\}$ follows from the definition of $S(\kappa)$ and the fact that $c \in \mathcal{C}_0$ if and only if $U^1(c) \in \mathcal{K}$. We first show that if $c \in \mathcal{E}(\mathcal{C}_0)$, $c \in \mathcal{C}$. As c satisfies R^H and $F\prime$ at all t, h_{t-1}, it satisfies P^1. At all t, h_{t-1} such that $U^1(c_{h_{t-1}}) = \kappa < \kappa^{**}, g(\kappa) < e$ by Lemma 5 and Step 1 of Lemma 12, and c satisfies R^L. At all t, h_{t-1} such that $U^1(c_{h_{t-1}}) = \kappa^{**}$, we proceed as in the proof to Lemma 6: At $b' = g(\kappa^{**}) - e, B' = G(\kappa^{**}), u_2(1-b') + \delta W(B') = U^2(c^*)$. Let b'' be defined by $u_1(b'') + \delta B' = u_1(b(\kappa^{**})) + \delta B(\kappa^{**})$; $b'' \leq b'$ would imply

$$u_2(1-b'') + \delta W(B') \geq u_2(1-b') + \delta W(B') = U^2(c^*),$$

which, as $B' > B(\kappa^{**})$, contradicts either uniqueness or optimality of the proposed solution $b(\kappa^{**}), B(\kappa^{**})$. Therefore, $b'' > b'$, and

$$u_1(b(\kappa^{**})) + \delta B(\kappa^{**}) = u_1(b'') + \delta B' > u_1(b') + \delta B' = u_1(g(\kappa^{**}) - e) + \delta G(\kappa^{**}).$$

Therefore c satisfies R^L at all t, h_{t-1}, implying that if $c \in S(\kappa)$ for $\kappa \in \mathcal{K}$, $c \in \mathcal{C}$.

Now suppose $c \in \mathcal{E}(\mathcal{C})$ and let $\kappa = U^1(c)$. As $\mathcal{C} \subset \mathcal{C}_0$, this implies $S(\kappa) \neq \varnothing$. Take $c' \in S(\kappa)$; by the above, $c' \subset \mathcal{C}$. Because c is efficient, it must be the case that $U^2(c) \geq U^2(c') = W(\kappa)$. If the inequality were strict, $\mathcal{C} \subset \mathcal{C}_0$ and $U^1(c) = U^1(c') = \kappa$ would contradict efficiency of c' in $\mathcal{C}_0$. Then it follows from $c \in \mathcal{C} \subset \mathcal{C}_0$ that $c \in S(\kappa)$, and from Lemma 2(ii) that $\{c\} = \{c'\} = S(\kappa)$ and therefore $c \in \mathcal{E}(\mathcal{C}_0)$.

On the other hand, let $c' \in \mathcal{E}(\mathcal{C}_0)$. Then $\{c'\} = S(\kappa')$ for some $\kappa' \in \mathcal{K}$; again, we have $c' \in \mathcal{C}$ by the first part of this proof. Take any $c \in \mathcal{C}$ such that $\kappa := U^1(c) \geq \kappa'$. If $\kappa = \kappa'$, $c \in \mathcal{C} \subset \mathcal{C}_0$ implies $W(\kappa') = U^2(c') \geq U^2(c)$. If $\kappa > \kappa', U^2(c') = W(\kappa') > W(\kappa) \geq U^2(c)$ by the same argument and the fact that W is strictly decreasing. In either case we conclude that $c' \in \mathcal{E}(\mathcal{C})$. ■

Proof of Proposition 1

Proposition 1 follows from Lemmas 2 and 14.■

8.3 Proof of Proposition 3

Claim 3 *If S holds, $b(\kappa) < 1$ for all κ.*

Proof. We have shown in Lemma 12 that $g(\kappa) \leq e$ for all κ. If S holds, this implies

$$\frac{u_2'(1 + e - g(\kappa))}{u_1'(g(\kappa))} \leq \frac{u_2'(1)}{u_1'(e)} \leq \frac{u_2'(0)}{u_1'(1)}$$

for all κ, which together with Proposition 2 implies that $b(\kappa) < 1$. ■

We first show $g(\kappa) < g(G(\kappa))$ for $\kappa < \kappa^R$.

Lemma 15 *g is a strictly increasing function.*

Proof. From Lemma 5 and $g(\kappa) \in (0, 1+e)$ (Lemma 12(i)), g is strictly increasing if γ is strictly increasing. In Step 1 of Lemma 12, we have shown γ to be strictly increasing on $[\kappa^P, \kappa^{**}]$. For $\kappa, \kappa' \in (\kappa^*, \kappa^P)$, $\kappa < \kappa'$, suppose $\gamma(\kappa) = \gamma(\kappa)$. This implies $b(\kappa) < b(\kappa')$, $B(\kappa) > B(\kappa')$. As Lemma 13 implies that $\partial V_H(\gamma) = V_H'(\gamma)$ for all γ and therefore $V_H'(\gamma(\kappa)) = V_H'(\gamma(\kappa'))$, we can again derive a contradiction from optimality and Step 2 of Lemma 10. ■

Lemma 16 *The functions g, G, b, B are continuous.*

Proof. g, G are continuous whenever γ is continuous. From Lemma 6, γ, b, B are continuous on $[\kappa^R, \kappa^{**}]$. For $\kappa \leq \kappa^R$, B, γ are continuous if b is. $b(\kappa) = b^P(\kappa)$ on $[\kappa^P, \kappa^R]$, which is continuous by continuity of u_i, W. For $\kappa < \kappa^P$, $b(\kappa)$ as solution to

$$\max_b \pi V_H(\kappa + (1-\pi)h(b)) + (1-\pi)V_L(b;\kappa) \text{ such that } b \leq z^{-1}(\kappa - \delta\kappa^*)$$

is continuous by strict concavity of u_i, W. The definition of κ^P in Lemma 10 implies $\lim_{\kappa \nearrow \kappa^P} b(\kappa) = b(\kappa^P)$. ■

Definition 10

$$\begin{aligned} q(\kappa) &= \frac{\pi u_1'(b(\kappa)+e)}{z'(b(\kappa))}, \\ \eta(\kappa, v_B) &= 1 + \frac{(1-\pi)h'(b(\kappa))}{\frac{u_2'(1-b(\kappa))}{v_B} + z'(b(\kappa))}, \ \textit{for } v_B \in \partial W(B(\kappa)). \end{aligned}$$

Lemma 17 *The subdifferential ∂W takes the form*

$$\partial W(\kappa) = \begin{cases} (V^H)'(\gamma(\kappa)), \ \textit{for } \kappa \in (\kappa^R, \kappa^{**}) \\ \{v | v = \pi\eta(\kappa, v_B)(V^H)'(\gamma(\kappa)), \}, \ \textit{for } \kappa \in (\kappa^P, \kappa^R) \\ -q(\kappa)\frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))} - (1-q(\kappa))\frac{u_2'(1-b(\kappa))}{u_1'(b(\kappa))}, \ \textit{for } \kappa \in (\kappa^*, \kappa^P) \end{cases}.$$

and $\partial W(\kappa) = [W_+'(\kappa), W_-'(\kappa)]$, for $\kappa \in \{\kappa^P, \kappa^R\}$.

Proof. As W is concave, $\partial W(\kappa) = [W_+'(\kappa), W_-'(\kappa)]$ at any κ. $W_+'(\kappa)$, $W_-'(\kappa)$ can be derived as

$$\begin{aligned} W_-'(\kappa) &= \lim_{\varepsilon \to 0} W_-'(\kappa - \varepsilon), \\ W_+'(\kappa) &= \lim_{\varepsilon \to 0} W_+'(\kappa + \varepsilon). \end{aligned}$$

We have shown in Lemma 6 that for $\kappa \in (\kappa^R, \kappa^{**})$,

$$W(\kappa) = \pi V^H(\frac{1}{\pi}[\kappa - (1-\pi)\beta(\kappa^{**})]) + (1-\pi)U^2(c^*),$$

which, as V^H is differentiable for $\kappa \in int\mathcal{K}$ by Lemma 13, shows the first part of our claim. For $\kappa \in (\kappa^P, \kappa^R)$, $B(\kappa)$, $b(\kappa)$, $\gamma(\kappa)$ are uniquely determined by

$$\begin{aligned} \kappa &= z(b(\kappa)) + \delta B(\kappa), \\ U^2(c^*) &= u_2(1-b(\kappa)) + \delta W(B(\kappa)), \\ \gamma(\kappa) &= \kappa + (1-\pi)h(b(\kappa)), \end{aligned}$$

and

$$W(\kappa) = \pi V^H(\gamma(\kappa)) + (1-\pi)U^2(c^*).$$

B is a continuous and strictly increasing function of κ; furthermore, at any κ such that W is differentiable at $B(\kappa), B, b$ are differentiable at κ. As W is strictly concave, there is only a countable number of points where it is not differentiable. By the Implicit Function Theorem, at a point κ such that W is differentiable,

$$b'(\kappa) = \frac{W'(B(\kappa))}{u_2'(1-b(\kappa)) + z'(b(\kappa))W'(B(\kappa))}.$$

Then for any κ, we can choose $\varepsilon > 0$ such that W is differentiable at $(B(\kappa), B(\kappa+\varepsilon)]$, and

$$W_+'(\kappa) = \lim_{\varepsilon \to 0} W_+'(\kappa+\varepsilon) = \lim_{\varepsilon \to 0} \pi(V^H)'(\gamma(\kappa+\varepsilon)) \left[1 + \frac{(1-\pi)h'(b(\kappa+\varepsilon))}{\frac{u_2'(1-b(\kappa+\varepsilon))}{W'(B(\kappa+\varepsilon))} + z'(b(\kappa+\varepsilon))} \right].$$

As u_i are everywhere differentiable, so are h, z, γ, as is V^H. Therefore, as B is strictly increasing on $[\kappa^P, \kappa^R]$,

$$\begin{aligned} W_+'(\kappa) &= \pi(V^H)'(\gamma(\kappa)) \left[1 + \frac{(1-\pi)h'(b(\kappa))}{\frac{u_2'(1-b(\kappa))}{\lim_{\varepsilon \to 0} W'(B(\kappa+\varepsilon))} + z'(b(\kappa))} \right] = \\ &= \pi(V^H)'(\gamma(\kappa)) \left[1 + \frac{(1-\pi)h'(b(\kappa))}{\frac{u_2'(1-b(\kappa))}{\lim_{B \searrow B(\kappa)} W'(B)} + z'(b(\kappa))} \right] = \\ &= \pi(V^H)'(\gamma(\kappa))\eta(\kappa, W_+'(B(\kappa)). \end{aligned}$$

Similarly we derive

$$W_-'(\kappa) = \pi(V^H)'(\gamma(\kappa))\eta(\kappa, W_-'(B(\kappa)),$$

and $\partial W(\kappa)$ is therefore as claimed, for $\kappa \in (\kappa^P, \kappa^R)$. If $\kappa < \kappa^P$ and $B(\kappa) > \kappa^*$, $(b(\kappa), B(\kappa)) \in (0,1) \times int\mathcal{K}$, and it is easy to see that

$$\begin{aligned} W(\kappa) = \max_b \pi V^H(u_1(b+e) + \delta B) + (1-\pi)[u_2(1-b) + \delta W(B)] \\ \text{s.t. } z(b) + \delta B = \kappa \end{aligned}$$

has a unique multiplier. u_i, W are strictly concave and bounded, and we conclude that W is differentiable, with

$$\partial W(\kappa) = -q(\kappa)\frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))} - (1-q(\kappa))\frac{u_2'(1-b(\kappa))}{u_1'(b(\kappa))}.$$

If $B(\kappa) = \kappa^*$,

$$W(\kappa) = \pi V^H(u_1(z^{-1}(\kappa - \delta\kappa^*) + e) + \delta\kappa^*) + (1-\pi)[u_2(1 - z^{-1}(\kappa - \delta\kappa^*)) + \delta W(\kappa^*)],$$

and indeed

$$\partial W(\kappa) = -q(\kappa)\frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))} - (1-q(\kappa))\frac{u_2'(1-b(\kappa))}{u_1'(b(\kappa))}.$$

■

By assumption, $\kappa < \kappa^R$. As γ strictly increasing by Lemma 15, G is weakly increasing by Lemma 5. We have shown $G(\kappa) = \kappa$ for $\kappa \in [\kappa^R, \kappa^{**}]$ in Step 5 of Lemma 12; therefore $\kappa < \kappa^R$ implies $G(\kappa) \leq G(\kappa^R) = \kappa^R$. We have also shown $G(\kappa) > \kappa^*$ for all κ in Lemma 12.

Case 1: $\kappa = \kappa^*$. The desired result then follows from Lemma 15 and $G(\kappa^*) > \kappa^*$.

Case 2: $\kappa > \kappa^*$, and $G(\kappa) = \kappa^R$. The result follows from $\kappa < \kappa^R$ and Lemma 15.

Case 3: $\kappa > \kappa^*, G(\kappa) \in (\kappa^*, \kappa^P)$.

Claim 1 and Lemma 13 imply that at any $\kappa \in (\kappa^*, \kappa^{**})$, there exists $v_G^e \in \partial W(G(\kappa))$ such that

$$\frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))} = -v_G^e.$$

As $G(\kappa) < \kappa^P$, we see from Lemma 17 that

$$\frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))} = q(G(\kappa))\frac{u_2'(1+e-g(G(\kappa)))}{u_1'(g(G(\kappa)))} + (1-q(G(\kappa)))\frac{u_2'(1-b(G(\kappa)))}{u_1'(b(G(\kappa)))},$$

which implies the desired result as $q(\kappa) < 1$ for all κ and

$$\frac{u_2'(1+e-g(G(\kappa)))}{u_1'(g(G(\kappa)))} > \frac{u_2'(1-b(G(\kappa)))}{u_1'(b(G(\kappa)))}$$

for all $G(\kappa)$ by Proposition 2.

Case 4: $\kappa > \kappa^*, G(\kappa) \in [\kappa^P, \kappa^R)$.

Claim 1, Lemmas 13 and 17 now imply

$$\frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))} = \pi\eta(G(\kappa), v_B)\frac{u_2'(1+e-g(G(\kappa)))}{u_1'(g(G(\kappa)))},$$

for some $v_B \in \partial W(B(G(\kappa)))$. We proceed to show that $\pi\eta(G(\kappa), v_B) < 1$ for all v_B. Note that in this case, $\kappa^P < \kappa^R$ by assumption. From the definition of κ^P, this implies $B(\kappa^R) > B(\kappa^P) \geq \kappa^*$. S implies $b(\kappa^R) < 1$, and we have $B(\kappa^R) < \kappa^{**}$ from Lemma 6,

$b(\kappa^R) > 0$ from $\kappa^R > \kappa^*$ and Step 1 of Lemma 8. The characterization of $b(\kappa^R)$, $B(\kappa^R)$ in Lemma 6 therefore imply that there exists $v_B^e \in \partial W(B(\kappa^R))$ such that

$$\frac{u_2'(1 - b(\kappa^R))}{u_1'(b(\kappa^R))} = -v_B^e.$$

For all $\kappa \in [\kappa^P, \kappa^R)$, $b(\kappa) > b(\kappa^R)$, $B(\kappa) < B(\kappa^R)$, and therefore

$$\frac{u_2'(1 - b(\kappa))}{u_1'(b(\kappa))} > -v_B, \text{ for all } v_B \in \partial W(B(\kappa)),$$

which implies $\eta(\kappa, v_B) < \frac{1}{\pi}$ for all $v_B \in \partial W(B(\kappa))$, and we are done.

We now show $b(\kappa) > b(B(\kappa))$ for $\kappa > \kappa^*$.

If $B(\kappa) = \kappa^*$, The result follows from $b(\kappa^*) = 0$ and Step 1 of Lemma 8. In the following, we therefore assume $B(\kappa) > \kappa^*$.

Step 1: At all κ such that $B(\kappa) > \kappa^*$,

$$\frac{u_2'(1 - b(\kappa))}{u_1'(b(\kappa))} \geq -W_-'(B(\kappa)).$$

Proof of Step 1:

Case 1: $\kappa \in [\kappa^R, \kappa^{**}]$. We have $b(\kappa) = b(\kappa^R) = b(\beta(\kappa^{**}); u^1, 1)$, $B(\kappa) = B(\kappa^R) = B(\beta(\kappa^{**}); u^1, 1)$ by Step 2 of Lemma 6. $b(\kappa^R) < 1$ and $B(\kappa) > \kappa^*$ then imply that there exists $v_B^e \in \partial W(B(\kappa^R))$ such that

$$\frac{u_2'(1 - b(\kappa^R))}{u_1'(b(\kappa^R))} \geq -v_B^e \geq -W_-'(B(\kappa^R)).$$

Case 2: $\kappa \in [\kappa^P, \kappa^R)$. By assumption $\kappa^P < \kappa^R$ and therefore $b(\kappa) > b(\kappa^R), B(\kappa) < B(\kappa^R)$, implying

$$\frac{u_2'(1 - b(\kappa))}{u_1'(b(\kappa))} > \frac{u_2'(1 - b(\kappa^R))}{u_1'(b(\kappa^R))} \geq -W_-'(B(\kappa^R)) > -W_-'(B(\kappa)).$$

Case 3: $\kappa \in (\kappa^*, \kappa^P)$. By claim 2, we have $b(\kappa) \geq b' := b(\beta(\kappa); u^1, 1)$, $B(\kappa) \leq B' := B(\beta(\kappa); u^1, 1)$. Therefore, $b(\kappa) < 1$ and $B(\kappa) > \kappa^*$ again imply, for $v_{B'}^e \in \partial W(B')$,

$$\frac{u_2'(1 - b(\kappa))}{u_1'(b(\kappa))} \geq \frac{u_2'(1 - b')}{u_1'(b')} \geq -v_{B'}^e \geq -W_-'(B(\kappa)).$$

□

Step 2: At all $\kappa > \kappa^*$,

$$-W_-'(\kappa) \geq q(\kappa)\frac{u_2'(1 + e - g(\kappa))}{u_1'(g(\kappa))} + (1 - q(\kappa))\frac{u_2'(1 - b(\kappa))}{u_1'(b(\kappa))}.$$

Proof of Step 2:

Case 1: $\kappa \in (\kappa^R, \kappa^{**}]$. The result follows from Lemma 17, Proposition 2 and $q(\kappa) < 1$.

Case 2: $\kappa \in (\kappa^P, \kappa^R]$.

The first order condition to $W(\kappa)$ and definition of κ^P in Lemma 10 imply, for all $v_B \in \partial W(B(\kappa))$,

$$-\pi V_H'(\gamma(\kappa)) - (1-\pi)v_B > q(\kappa)\frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))} + (1-q(\kappa))\frac{u_2'(1-b(\kappa))}{u_1'(b(\kappa))}.$$

It is easy to see that then

$$\eta(\kappa, v_B) - 1 > \frac{1-\pi}{\pi}\frac{v_B}{V_H'(\gamma(\kappa))}.$$

Hence

$$\begin{aligned} W_-'(\kappa) &= -\pi\eta(\kappa, W_-'(B(\kappa)))V_H'(\gamma(\kappa)) > -\pi V_H'(\gamma(\kappa)) - (1-\pi)W_-'(B(\kappa) > \\ &> q(\kappa)\frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))} + (1-q(\kappa))\frac{u_2'(1-b(\kappa))}{u_1'(b(\kappa))}. \end{aligned}$$

for all $v_B \in \partial W(B(\kappa))$.

Case 3: $\kappa \in (\kappa^*, \kappa^P]$. The result follows from Lemma 17. $\square$

The desired result now follows from Steps 1 and 2, $B(\kappa) > \kappa^*$ and Proposition 2.

$g(\kappa) > f(b(G(\kappa))$ follows from the fact that

$$\frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))} \geq -v_G^e \geq -W_-'(G(\kappa))$$

at all κ, Step 2 of the proof of $b(\kappa) > b(B(\kappa)$ and $G(\kappa) > \kappa^*$ for all κ and Proposition 2.$\blacksquare$

Lemma 18 *For* $\kappa = \kappa^*$,

$$\frac{u_1'(g(\kappa))}{u_1'(b(\kappa))} < \frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))}.$$

Proof. As $g(\kappa^*) < e$ and $\kappa^* < G(\kappa^*) \leq G(\kappa^R) < \kappa^{**}$ by Lemma 12, by Claim 1 and Step 2 of Proposition 3 there exists $v_G^e \in \partial W(G(\kappa^*))$ such that

$$\frac{u_2'(1+e-g(\kappa^*))}{u_1'(g(\kappa^*))} \geq v_G^e \geq q(G(\kappa^*))\frac{u_2'(1+e-g(G(\kappa^*)))}{u_1'(g(G(\kappa^*)))} + (1-q(G(\kappa^*)))\frac{u_2'(1-b(G(\kappa^*)))}{u_1'(b(G(\kappa^*)))}.$$

Using Proposition 2 and $b(G(\kappa^*)) > b(\kappa^*) = 0$, this implies

$$\frac{u_2'(1+e-g(\kappa^*))}{u_1'(g(\kappa^*))} > \frac{u_2'(1-b(G(\kappa^*)))}{u_1'(b(G(\kappa^*)))} > \frac{u_2'(1-b(\kappa^*))}{u_1'(b(\kappa^*))}.$$

$\blacksquare$

8.4 Proof of Proposition 4

Lemma 19 *If* $G(\kappa^*) < \kappa^P$, G *is strictly increasing on* $[\kappa^*, \min \kappa : G(\kappa) = \kappa^P]$.

Proof. G is continuous by Lemma 16, $G(\kappa^*) < \kappa^P$, and $G(\kappa^R) = \kappa^R \geq \kappa^P$ The minimum above is therefore well defined. From the first order condition in Lemma 5, it is easy to see that G increases strictly wherever $W(G(\kappa))$ is differentiable. The result then follows from Lemma 17. ■

Lemma 20 *If* $B(\kappa^R) > \kappa^*$ *and either* C *or* L *hold,* B *is strictly increasing on* $[\max \kappa : B(\kappa) = \kappa^*, \kappa^R]$.

Proof. We have shown in Proposition 5 that $\kappa^P < \kappa^R$ whenever $B(\kappa^R) > \kappa^*$. B strictly increasing on $[\kappa^P, \kappa^R]$ is a consequence of R^H and P_L^2 for $\kappa \in [\kappa^P, \kappa^R]$. If κ^1 is defined, B is strictly increasing on $[\kappa^1, \kappa^P]$ by R^H and the adding up constraint. We now show that (L) or (C) imply that B is strictly increasing on $[\max \kappa : B(\kappa) = \kappa^*, \min\{\kappa^1, \kappa^P\}]$.

Step 1: If $B(\kappa) \geq B(\kappa')$ for $\kappa < \kappa' < \kappa^P$,

$$\frac{u_1'(g(\kappa'))}{u_1'(b(\kappa'))} > \frac{u_2'(g(\kappa))}{u_2'(b(\kappa))} \text{ and } \frac{u_2'(1+e-g(\kappa'))}{u_2'(1-b(\kappa'))} < \frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))}.$$

Proof of Step 1: $\kappa < \kappa' < \kappa^P$, $b(\kappa) < b(\kappa')$, $g(\kappa) < g(\kappa')$ and $G(\kappa) \leq G(\kappa')$ by Proposition 5 and Lemmas 15 and 19. Then $B(\kappa) \geq B(\kappa')$ and R^H imply that

$$u_1(g(\kappa')) - u_1(g(\kappa)) \leq u_1(b(\kappa') + e) - u_1(b(\kappa) + e).$$

It follows from strict concavity of u_1 and $g(\kappa) < b(\kappa) + e$ for all κ that $g(\kappa') - g(\kappa) < b(\kappa') - b(\kappa)$. Then define $\Delta' = g(\kappa') - b(\kappa')$, $\Delta = g(\kappa) - b(\kappa)$. Then

$$\frac{u_1'(g(\kappa'))}{u_1'(b(\kappa'))} > \frac{u_2'(b(\kappa') + \Delta)}{u_2'(b(\kappa'))} \geq \frac{u_2'(b(\kappa) + \Delta)}{u_2'(b(\kappa))} = \frac{u_2'(g(\kappa))}{u_2'(b(\kappa))},$$

where the second inequality follows from Lemma 9. Similarly, we show

$$\frac{u_2'(1+e-g(\kappa'))}{u_2'(1-b(\kappa'))} < \frac{u_2'(1+e-g(\kappa))}{u_2'(1-b(\kappa))}.$$

□

Step 2: If $B(\kappa) \geq B(\kappa')$ for $\kappa < \kappa' < \kappa^P$ and either C or L hold,

$$\frac{u_1'(b(\kappa) + e)}{u_1'(g(\kappa))} \geq \frac{u_1'(b(\kappa') + e)}{u_1'(g(\kappa'))}.$$

Proof of Step 2: If C holds,

$$\frac{u_1'(b(\kappa')+e)}{u_1'(b(\kappa)+e)} = \frac{u_1'(b(\kappa'))}{u_1'(b(\kappa))} < \frac{u_1'(g(\kappa'))}{u_1'(g(\kappa))}.$$

by Step1. Suppose L is true. From Step 1, we have $u_1(g(\kappa')) - u_1(g(\kappa)) \leq u_1(b(\kappa')+e) - u_1(b(\kappa)+e)$. It is easy to see that this implies the desired result if $u^1(c) = \ln(a+dc)$.□

We can now write the first order condition to $W(\kappa)$, if $B(\kappa) > \kappa^*$ and $\kappa < \kappa^P$, as

$$\frac{z'(b(\kappa))}{u_1'(b(\kappa)+e)} = \frac{-\pi V_H'(\gamma(\kappa)) + (1-\pi)\frac{u_2'(1-b(\kappa))}{u_1'(b(\kappa)+e)}}{-\pi V_H'(\gamma(\kappa)) - (1-\pi)W'(B(\kappa))}.$$

Suppose $B(\kappa) \geq B(\kappa') > \kappa^*$ for $\kappa < \kappa' < \kappa^P$. Using Lemma 9 and $b(\kappa) < b(\kappa')$, we have

$$\frac{z'(b(\kappa))}{u_1'(b(\kappa)+e)} \geq \frac{z'(b(\kappa'))}{u_1'(b(\kappa')+e)}.$$

As the first order condition holds with equality at both κ, κ', this implies

$$-W'(B(\kappa))[\pi(\frac{u_2'(1+e-g(\kappa'))}{u_1'(g(\kappa'))} - \frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))}) + (1-\pi)(\frac{u_2'(1-b(\kappa'))}{u_1'(b(\kappa')+e)} - \frac{u_2'(1-b(\kappa))}{u_1'(b(\kappa)+e)}) +$$
$$+\pi[\frac{u_2'(1+e-g(\kappa))}{u_1'(g(\kappa))}\frac{u_2'(1-b(\kappa'))}{u_1'(b(\kappa')+e)} - \frac{u_2'(1+e-g(\kappa'))}{u_1'(g(\kappa'))}\frac{u_2'(1-b(\kappa))}{u_1'(b(\kappa)+e)}] \leq 0.$$

But as $g(\kappa) < g(\kappa')$, $b(\kappa) < b(\kappa')$, and by Steps 1 and 2, it is easy to see that the expression above is strictly positive if $B(\kappa) \geq B(\kappa')$. ■

Claim 4 *For all κ, $G(\kappa) > B(\kappa)$. For $\kappa \in (\kappa^*, \kappa^R)$, $B(\kappa) < \kappa < G(\kappa)$.*

Proof. For $\kappa \in [\kappa^*, \kappa^R]$, R^H holds and $g(\kappa) < e$ by Lemmas 12 and 15, implying $g(\kappa) < b(\kappa) + e$ and therefore $G(\kappa) > B(\kappa)$ by R^H. For $\kappa > \kappa^R$, $G(\kappa) = \kappa > \kappa^R > B(\kappa^R) \geq B(\kappa)$ by Step 5 of Lemma 12, Proposition 3 and Step 2 of Lemma 8. $B(\kappa) < \kappa$ for $\kappa \in (\kappa^*, \kappa^R)$ follows from $B(\kappa^R) < \kappa^P$, Proposition 3 and Proposition 5. $G(\kappa) < \kappa$ for $\kappa \in (\kappa^*, \kappa^R)$ follows from Lemma 15 and Proposition 3. ■

We shall prove the desired result for histories h_t, h_t' such that $\#h_t = \#h_t' + 1$. This is sufficient as we can construct any more fortunate history h_t from h_t' by switching high-income state occurences one at a time and applying the result. Let h_t be more fortunate after τ for agent 1 than h_t', with $h_t \in \mathcal{H}_t^+(\kappa)$, $h_t' \in \mathcal{H}_t^-(\kappa)$, for some $\kappa \in \mathcal{K}$. Then $s_{\tau+1} = H, s_{\tau+1}' = L$, and $s_k = s_k'$ otherwise, for $s_k \in h_t, s_k' \in h_t'$. Construct the abbreviated history $h_n = (s_1, ..., s_n)$ from h_t, such that $h_t = (h_n, s_{n+1}, ..., s_t)$, and define h_n' in the same way from h_t'.

We show that $U^1(c_{h_t}) > U^1(c_{h'_t})$, which implies the desired result by Lemma 15 and Proposition 5. The proof is by induction. Define $\kappa_n = U^1(c_{h_n}), \kappa'_n = U^1(c_{h'_n})$ for $n = \tau, ...t$. Note that

$$\kappa_{n+1} = \begin{cases} G(\kappa_n), \text{ if } s_n = H \\ B(\kappa_n), \text{ if } s_n = L \end{cases},$$

where $s_n \in h_t$. The same is of course true for κ'_n. Therefore, $\kappa_\tau = \kappa'_\tau$; and as $G(\kappa) > B(\kappa)$ for all κ by Claim 4, $\kappa_{\tau+1} > \kappa'_{\tau+1}$. As $h_t \in \mathcal{H}_t^+(\kappa)$, $h'_t \in \mathcal{H}_t^-(\kappa)$, $\kappa_n > \kappa^*$, $\kappa'_n < \kappa^P \leq \kappa^R$ by assumption, for $n \in \{\tau, ..., t\}$.

Now let $\kappa_n > \kappa'_n$, for $n \in \{\tau + 1, ...t\}$. We will show that then either $\kappa_{n+1} = \kappa'_{n+1} = \kappa^*$, $\kappa'_{n+1} = \kappa_{n+1} = \kappa^R$, or $\kappa_{n+1} > \kappa'_{n+1}$.

Case 1: $s_n = H$.

If $\kappa_n \geq \kappa^R$, $\kappa_{n+1} = \kappa_n \geq \kappa^P > \kappa'_{n+1}$ by Step 5 of Lemma 12 and $h'_t \in \mathcal{H}_t^-(\kappa)$. If $\kappa^R > \kappa_n \geq \kappa^P$, $\kappa_{n+1} > \kappa_n \geq \kappa^P > \kappa'_{n+1}$ by Lemma 4 and $h'_t \in \mathcal{H}_t^-(\kappa)$. If $\kappa_n < \kappa^P$, $\kappa_{n+1} > \kappa'_{n+1}$ by Lemma 19 (note that if $G(\kappa^*) > \kappa^P$, $s_n = H \notin h'_t \in \mathcal{H}_t^-(\kappa)$).

Case 2: $s_n = L$.

If $B(\kappa^{**}) = \kappa^*$, $B(\kappa) = \kappa^*$ for all κ by Step 2 of Lemma 8, and $s_n = L \notin h_t \in \mathcal{H}_t^+(\kappa)$. We therefore proceed with the case $B(\kappa^{**}) > \kappa^*$.

If $\kappa_n \geq \kappa^R, \kappa_{n+1} = B(\kappa^{**}) > \kappa'_{n+1}$ by Lemma 20 and $\kappa_n \geq \kappa^R > \kappa'_n$. If $\kappa_n < \kappa^R$, $\kappa_{n+1} > \kappa'_{n+1}$ by Lemma 20 and $h_t \in \mathcal{H}_t^+(\kappa)$.■

8.5 Proof of Proposition 5

We have already shown that P_L^2 binds at all $\kappa \geq \kappa^P$ in Lemma 10.

Lemma 21 *b is strictly increasing on $[\kappa^*, \kappa^P)$.*

Proof. Suppose $b(\kappa) \geq b(\kappa')$, for $\kappa < \kappa' < \kappa^P$. Then $B(\kappa) < B(\kappa')$, and $\gamma(\kappa) < \gamma(\kappa')$ by Lemma 15. We can again derive a contradiction from Step 2 of Lemma 10. ■

For $\kappa \in [\kappa^P, \kappa^R]$, $b(\kappa) = b^P(\kappa)$ which is strictly decreasing by Lemma 8.

As $0 < b(\kappa^R) = b(\kappa^{**}) < 1$ and $\kappa^* < B(\kappa^R) = B(\kappa^{**}) < \kappa^{**}$ by assumption and $(b(\kappa^{**}), B(\kappa^{**}))$ solve $V(\beta(\kappa^{**}); u_1, 1)$, we must have

$$-W'(B(\kappa^R)) = \frac{u'_2(1 - b(\kappa^R))}{u'_1(b(\kappa^R))},$$

and therefore

$$V'(b(\kappa^R); \kappa^R) = (-\pi h'(b(\kappa^R)))[\frac{u'_2(1 + e - g(\kappa^R))}{u'_1(g(\kappa^R))} - \frac{u'_2(1 - b(\kappa^R))}{u'_1(b(\kappa^R))}] > 0.$$

But if $\kappa^P = \kappa^R$, continuity of g, b, G, B (Lemma 16) and V' must imply $V'(b(\kappa^R); \kappa^R) = 0$. Therefore $\kappa^P < \kappa^R$. ■

8.6 Proof of Proposition 6

(i) For ease of notation, we write $\{\kappa_t\}$ for the process $\{U^1(c_{h_t})\}$ in this proof; the history dependence of $\{\kappa_t\}$ is implicit. G, B are measurable functions. Then the process $\{\kappa_t\}$ is a Markov process; also the inverses $G^{-1}(K) = \{\kappa \in \mathcal{K} | G(\kappa) = \kappa', \kappa' \in K\}$, $B^{-1}(K)$, for $K \in \mathcal{B}(\mathcal{K})$, are well defined. Define the function $\Pi : \mathcal{K} \times \mathcal{B}(\mathcal{K}) \to [0, 1]$ by

$$\Pi(\kappa, K) = \left\{ \begin{array}{c} 1, \kappa \in G^{-1}(K) \cap B^{-1}(K) \\ \pi, \kappa \in G^{-1}(K) \cap (B^{-1}(K))^c \\ 1 - \pi, \kappa \in (G^{-1}(K))^c \cap B^{-1}(K) \\ 0, \kappa \in (G^{-1}(K) \cup B^{-1}(K))^c \end{array} \right. .$$

Π is a markovian kernel and defines transition probabilities for the process $\{\kappa_t\}$. n-step transition probabilities $\Pi^n(\kappa, K)$ are defined in the obvious way, for $n = 1, 2, ...$ Define the adjoint $T^* : M(\mathcal{B}(\mathcal{K}), \mathcal{K}) \to M(\mathcal{B}(\mathcal{K}), \mathcal{K})$ of Π by

$$(T^*\mu)(K) = \int_{\mathcal{K}} \Pi(\kappa, K) d\mu, \text{ for all } K \in \mathcal{B}(\mathcal{K}),$$

so that we can define $\mu_t(\mu_0)$ inductively by $\mu_t(\mu_0) = T^*\mu_{t-1}(\mu_0), t = 1, 2, ...$ Convergence is in the total variation norm. Using Theorem 11.12 of Stokey and Lucas (1989), the following condition is necessary and sufficient for strong convergence of the Markov process:

$$\text{There exist } \varepsilon > 0 \text{ and an integer } N \geq 1 \text{ such that for any } K \in \mathcal{B}(\mathcal{K}), \qquad (M)$$
$$\text{either } \Pi^N(\kappa, K) \geq \varepsilon, \forall \kappa \in \mathcal{K}, \text{ or } \Pi^N(\kappa, K^c) \geq \varepsilon, \forall \kappa \in \mathcal{K}.$$

This implies the existence of a unique $\bar{\mu} \in M(\mathcal{B}(\mathcal{K}), \mathcal{K})$ such that

$$||\mu_{Nt}(\mu_0) - \bar{\mu}|| \leq (1 - \varepsilon)^t ||\mu_0 - \bar{\mu}||, t = 1, 2, ..., \text{ for all } \mu_0 \in M(\mathcal{B}(\mathcal{K}), \mathcal{K}),$$

or $\mu_t(\mu_0) \to \bar{\mu}$ strongly.

Claim 5 *There exists* $\kappa^B > \kappa^* : B(\kappa) = \kappa^*, \kappa \in [\kappa^*, \kappa^B]$.

Proof. If $B(\kappa^{**}) = \kappa^*$, we define $\kappa^B = \kappa^{**}$, and we are done by Step 2 of Lemma 8. We therefore proceed with the case $B(\kappa^R) > \kappa^*$. We first show that there exists

$\kappa^B > \kappa^* : B(\kappa) = \kappa^*, \kappa \in [\kappa^*, \kappa^B]$. Suppose not. Then, as $\kappa^* < \kappa^P$, there must exist $\Delta_1 > 0 : \partial V(b(\kappa); \kappa) = 0$, $\kappa \in (\kappa^*, \kappa^* + \Delta_1]$, and by continuity of c^κ and differentiability of u_i and $W(B(\kappa))$ (note $B(\kappa) < \kappa^P, \forall \kappa$), we must have $\lim_{\Delta_1 \to 0} \partial V(b(\kappa^* + \Delta_1); \kappa^* + \Delta_1) = 0$. We have

$$\begin{aligned}
&\lim_{\Delta_1 \to 0} \partial V(b(\kappa^* + \Delta_1); \kappa^* + \Delta_1) = \\
&\qquad = \pi \lim_{\Delta_1 \to 0} (-h'(b(\kappa^* + \Delta_1))(-V_H'(\gamma(\kappa^* + \Delta_1))) - u_2'(1) - \\
&\qquad - \lim_{\Delta_1 \to 0} z'(b(\kappa^* + \Delta_1)) W'(B(\kappa^* + \Delta_1)).
\end{aligned}$$

As $b(\kappa^*) = 0$, we need to treat the case where u_1 satisfies the Inada condition $\lim_{c \to 0} u_1'(c) = \infty$ separately from other specifications. First, suppose the Inada condition does not hold. Then, using

$$\begin{aligned}
W_+'(B(\kappa^*)) &= W_+'(\kappa^*) = q(\kappa^*) m^g(g(\kappa^*)) + (1 - q(\kappa^*)) m^b(b(\kappa^*)) = \\
&= \frac{\pi u_1'(b(\kappa^*) + e)}{z'(b(\kappa^*))} \frac{u_2'(1 + e - g(\kappa^*))}{u_1'(g(\kappa^*))} + \frac{(1-\pi) u_1'(b(\kappa^*))}{z'(b(\kappa^*))} \frac{u_2'(1 - b(\kappa^*))}{u_1'(b(\kappa^*))},
\end{aligned}$$

it is easy to see that

$$\lim_{\Delta_1 \to 0} \partial V(b(\kappa^* + \Delta_1); \kappa^* + \Delta_1) = \pi u_1'(b(\kappa^*))[m^g(g(\kappa^*)) - m^b(b(\kappa^*))] > 0$$

by Proposition 2.

If the Inada condition holds for u_1, note that $\lim_{\Delta_1 \to 0}(-h'(b(\kappa^* + \Delta_1)) = \infty$, but also $\lim_{\Delta_1 \to 0} z'(b(\kappa^* + \Delta_1)) = \infty$, while $\lim_{\Delta_1 \to 0}(-W'(B(\kappa^* + \Delta_1))) = 0$. However, $\lim_{\Delta_1 \to 0} z'(b(\kappa^* + \Delta_1))(-W'(B(\kappa^* + \Delta_1))) \geq 0$ for all Δ_1, and therefore again $\lim_{\Delta_1 \to 0} \partial V(b(\kappa^* + \Delta_1); \kappa^* + \Delta_1) > 0$. $\blacksquare$

To show M, it suffices to show that there exists $\varepsilon > 0$ and an integer $N \geq 1$ such that $\Pi^N(\kappa, \kappa^*) \geq \varepsilon$, for all $\kappa \in \mathcal{K}$. Then for any $K \in \mathcal{B}(\mathcal{K})$, either $\kappa^* \in K$ and therefore $\Pi^N(\kappa, K) \geq \varepsilon, \forall \kappa \in \mathcal{K}$, or $\kappa^* \in K^c$, and $\Pi^N(\kappa, K^c) \geq \varepsilon, \forall \kappa \in \mathcal{K}$.

Case 1: $B(\kappa^{**}) = \kappa^*$. Then $B(\kappa) = \kappa^*$ for all κ, and we define $\varepsilon = 1 - \pi$, $N = 1$.

Case 2: $B(\kappa^{**}) > \kappa^*$.

Define $\Delta_2 = \min \left[\kappa^B - \kappa^*, \min_{\kappa \in [\kappa^B, \kappa^{**}]}(\kappa - B(\kappa))\right]$. Δ_2 is well defined as B is continuous by Lemma 16. By Claim 5 and Proposition 3, which implies $B(\kappa) < \kappa$ for $\kappa > \kappa^B$, we have $\Delta_2 > 0$. For all $h_{t-1} \in \mathcal{H}_{t-1} \cup \{\varnothing\}$ and $h_t' = (h_{t-1}, L)$, $U^1(c_{h_t'}) \leq$

$U^1(c_{h_{t-1}}) - \Delta_2$. Define N, ε by

$$\begin{aligned} N &= 1 + \left[\min n \in \mathbb{N} : n \geq \frac{\kappa^{**} - \kappa^B}{\Delta_2} \right] \\ \varepsilon &= (1-\pi)^N. \end{aligned}$$

Then, as $B(\kappa^*) = \kappa^*$, we have that for any κ_t, $\kappa_{t+N-1} \in [\kappa^*, \kappa^B]$ if state L occured $N-1$ times in succession after κ_t, and therefore $\kappa_{t+N} = \kappa^*$ if L occurs one more time. The value function is bounded, so that N is finite. As N is finite, $\varepsilon > 0$.

(ii) Define a consequent set of κ as the set K' such that $\Pi^n(\kappa, K') = 1$, $n = 1, 2, ...$ An invariant set K' is a consequent set of all $\kappa \in K'$. As $\bar{\mu}$ is unique, the ergodic set $\bar{\mathcal{K}}$ is unique, and is the smallest invariant set, justifying our definition of $\bar{\mathcal{K}} = \{\kappa \in \mathcal{K} | \bar{\mu}(\kappa) > 0\}$. As $B(\kappa) < \kappa^P \leq \kappa^R$, and $G(\kappa) < \kappa^R$ for all κ, $[\kappa^*, \kappa^R]$ is an invariant set. Any invariant set contains an ergodic set; by (i), the ergodic set is unique, and therefore $\bar{\mathcal{K}} \in [\kappa^*, \kappa^R]$. By the definition of $\bar{\mathcal{K}}$, if $\kappa \in \bar{\mathcal{K}}$ and $\Pi^N(\kappa, \kappa') > 0$ for some finite integer N and $\kappa' \in \mathcal{K}$, we must also have $\kappa' \in \bar{\mathcal{K}}$. As we have shown in (i) that there exists a finite N such that $\Pi^N(\kappa, \kappa^*) \geq \varepsilon$, for all $\kappa \in \mathcal{K}$, $\kappa^* \in \bar{\mathcal{K}}$. If there exists $\kappa < \kappa^R$ such that $G(\kappa) = \kappa^R$, we can use the same technique as in (i) to show that $\kappa^R \in \bar{\mathcal{K}}$. If not, $G(\kappa) < \kappa^R$ for all $\kappa < \kappa^R$. Fix some $\kappa \in \bar{\mathcal{K}}$, and let κ^n be defined by $\kappa^n = G(\kappa^{n-1})$, $n = 1, 2, ...$, $\kappa_0 = \kappa$. For any finite n, $\kappa^n \in \bar{\mathcal{K}}$, and $\kappa^R > \kappa^n > \kappa^{n-1}$. But as $G(\kappa^R) = \kappa^R$ and G is continuous, $\lim_{n\to\infty} \kappa^n = \kappa^R$, and therefore $\kappa^R \in cl(\bar{\mathcal{K}})$.

(iii) If $B(\kappa^{**}) > \kappa^*$, $\kappa^R > \kappa^P$ from Proposition 5, B is strictly increasing on $[\kappa^P, \kappa^R]$ and therefore $B(\kappa) > \kappa^*$ for all $\kappa > \kappa^P$. Using the methods from (ii), there exists a finite N and an $\varepsilon' > 0$ such that $\Pi^N(\kappa, (\kappa^P, \kappa^R)) \geq \varepsilon'$ for all κ, showing that there must exist $\kappa' \in (\kappa^P, \kappa^R)$ such that $\kappa' \in \bar{\mathcal{K}}$. But then $B(\kappa') > \kappa^*$ is also in $\bar{\mathcal{K}}$.■

8.7 Proof of Proposition 7

Claim 6 *For $n = 1, 2, ...$, if $\mathcal{K}_{n-1}$ is compact and W_{n-1} is strictly concave, $\mathcal{K}_n$ is compact and W_n is strictly concave.*

Proof. If $\mathcal{K}_{n-1}$ is compact and W_{n-1} is strictly concave, the maximum $W_n(\kappa)$ is attained for all $\kappa \in \mathcal{K}_{n-1}$, and κ_n^{**} is again defined by $W_n(\kappa_n^{**}) = U^2(c^*)$. To show strict concavity of W_n, let (g', b', G', B') solve $W_n(\kappa_n')$, and (g'', b'', G'', B'') solve $W_n(\kappa_n'')$, with $\kappa_n' < \kappa_n''$, so that at least one element of (g', b', G', B') is distinct from (g'', b'', G'', B''). Then for any $\lambda \in (0,1)$, define g, b, G, B by

$$\begin{aligned} u_1(g) &= \lambda u_1(g') + (1-\lambda)u_1(g''), \\ u_1(b) &= \lambda u_1(b') + (1-\lambda)u_1(b''), \\ G &= \lambda G' + (1-\lambda)G'', \\ B &= \lambda B' + (1-\lambda)B''. \end{aligned}$$

Clearly, (g, b, G, B) satisfies P^2 and F', and R^H by nonincreasing absolute risk aversion of u_1 (see proof of Lemma 1); and by strict concavity of u_2, W_{n-1},

$$\begin{aligned} W_n(\lambda\kappa_n' + (1-\lambda)\kappa_n'') &\geq \pi[u_2(1+e-g) + \delta W_n(G)] + (1-\pi)[u_2(1-b) + \delta W_n(B)] \\ &> \lambda W_n(\kappa_n') + (1-\lambda)W_n(\kappa_n''). \end{aligned}$$

It is easily seen that W_0 is strictly concave under our assumptions on u_i. Furthermore, W_0 takes values on $\mathcal{K}_0 = [\kappa^*, \kappa_0^{**}]$, where κ_0^{**} is defined in familiar fashion by $W_0(\kappa_0^{**}) = U^2(c^{**})$. Note that $\mathcal{K}_0$ is compact; therefore $W_n, \mathcal{K}_n$ are well defined for all n. ■

We first extend W_n to the interval $\mathcal{K}_0$: define

$$\bar{W}_n(\kappa) = \begin{cases} W_n(\kappa), \kappa \in \mathcal{K}_n \\ U^2(c^*), \kappa \in \mathcal{K}_0 \backslash \mathcal{K}_n \end{cases}.$$

We define $\bar{W}$ in the same way. W_0 can be written

$$W_0(\kappa) = \max U^2(c) \text{ subject to } P^1, P^2, F \text{ and } U^1(c) \geq \kappa.$$

We first show that there exists a function $W_\infty(\kappa) \geq W$ such that $\{W_n\}_{n=1}^\infty$ converges uniformly to W_∞ on $\{\mathcal{K}_n\}_{n=1}^\infty$.

It is easy to see that $W_0 \geq W$ and $\kappa_0^{**} \geq \kappa^{**}$. If F is a space of functions on $\mathcal{K}_0$, we define the operator T:$F \to F$ by

$$Tf = \max \pi[u_2(1+e-g) + \delta f(G)] + (1-\pi)[u_2(1-b) + \delta f(B)]$$

$$\text{subject to } u_1(g) + \delta G \geq u_1(b+e) + \delta B, \tag{R^H}$$

$$u_2(1+e-g) + \delta f(G) \geq U^2(c^*), \tag{P_H^2}$$

$$u_2(1-b) + \delta f(B) \geq U^2(c^*), \tag{P_L^2}$$

$$(g, b, G, B) \in [0, 1+e] \times [0,1] \times \mathcal{K}_0^2, \text{ and} \tag{F'}$$

This operator is monotone, and one of its fixpoints is $\bar{W}$. Therefore, $\bar{W}_1 = T\bar{W}_0 \geq T\bar{W} = \bar{W}$,and repeated applications of T yield $\bar{W} \leq \bar{W}_n \leq \bar{W}_{n-1}$, for all n. Then for any $\kappa \in \mathcal{K}_0$, the sequence $\{\bar{W}_n(\kappa)\}_{n=1}^{\infty}$ is monotone and bounded below, so that its limit exists. We can then define the function $\bar{W}_\infty$ by

$$\bar{W}_\infty(\kappa) = \lim_{n\to\infty}\{\bar{W}_n(\kappa)\}_{n=1}^{\infty}, \kappa \in \mathcal{K}_0,$$

and $\{\bar{W}_n\}$ converges pointwise to $\bar{W}_\infty$. But then by compactness of $\mathcal{K}_0$, continuity of $\bar{W}_n$ and $\bar{W}_n \leq \bar{W}_{n-1}$, for all n, $\{\bar{W}_n\}$ also converges uniformly to $\bar{W}_\infty$ on $\mathcal{K}_0$. Note that $\bar{W}_\infty \geq \bar{W}$, and $\bar{W}_\infty$ continuous by continuity of $\bar{W}_n$. Therefore we can define $\kappa_\infty^{**} = \max \kappa : \bar{W}_\infty(\kappa) \geq U^2(c^*)$, where $\kappa_\infty^{**} \geq \kappa^{**} > \kappa^*$. Then $\lim_{n\to\infty}\{\mathcal{K}_n\}_{n=1}^{\infty} = \mathcal{K}_\infty := [\kappa^*, \kappa_\infty^{**}]$. For any $m = 1, 2, ...,$ we can restrict $\{\bar{W}_n\}$ to $\mathcal{K}_m$, and as $\mathcal{K}_m$ compact for all m, $\{\bar{W}_n\}$ also converges uniformly to the restriction of $\bar{W}_\infty$ on $\mathcal{K}_m$. Therefore $\{\bar{W}_n\}$ converges uniformly to $\bar{W}_\infty$ on $\{\mathcal{K}_n\}$, which is equivalent to $\{W_n\}$ converging uniformly to W_∞ on $\{\mathcal{K}_n\}$, where W_∞ is the restriction of $\bar{W}_\infty$ to $\mathcal{K}_\infty$.$\Box$

It only remains to be shown that $W_\infty = W$. Define the set $\mathcal{W}$ by

$$\mathcal{W} = \{(w^1, w^2) | \exists c \text{ s.t } w^1 = U^1(c), w^2 = U^2(c), \text{ and } c \text{ satisfies } P, R, F\}.$$

In the following, a tuple $(g, b) \in [0, 1+e] \times [0, 1]$ will again refer to agent 1's state-dependent consumption. For some compact set $\mathcal{A} \subseteq \mathbb{R}^2$, we draw continuation values $A := (A_H, A_L) \in \mathcal{A}^2$ for the agents. We can then write agents' expected utility in vector form as

$$U(g, b, A) = \begin{bmatrix} \pi u_1(g) + (1-\pi)u_1(b) + \delta[\pi A_H^1 + (1-\pi)A_L^1] \\ \pi u_2(1+e-g) + (1-\pi)u_2(1-b) + \delta[\pi A_H^2 + (1-\pi)A_L^2] \end{bmatrix},$$

so that we can define the operator Q by

$$Q(\mathcal{A}) = \{U(g, b, A) | A \in \mathcal{A}^2 \text{ and } (g, b, A) \text{ satisfy } P, R, F\}.$$

Following Abreu, Pearce and Stacchetti's (1990) Theorem 1, we show that for a compact set $\mathcal{A}$, if $\mathcal{A} \subseteq Q(\mathcal{A}), \mathcal{A} \subseteq \mathcal{W}$. For any $a \in \mathcal{A}$, construct the sharing rule $c(a)$ as follows: for $h_0 = \varnothing$, as $a \in Q(\mathcal{A})$, there exists g, b, A such that $a = U(g, b, A)$, and g, b, A satisfy P, R, F'. Set $c(a)(H) = g, c(a)(L) = b$. But again, as $A_H \in \mathcal{A} \subseteq Q(\mathcal{A})$, we can do the same construction for $h_1 = H$; and similarly for $h_1 = L$ as $A_L \in \mathcal{A} \subseteq Q(\mathcal{A})$. By induction on the event tree, the sharing rule $c(a)$ therefore satisfies P, R, F' at all t, h_t by construction, and $U^1(c(a)) = a^1, U^2(c(a)) = a^2$. Therefore $a = (a^1, a^2) \in \mathcal{W}$.

As $W_\infty \geq W$ and W defines the upper bound of $\mathcal{W}$, we are done if we can show that $graph W_\infty \subseteq Q(graph W_\infty)$. For any $w \in graph W_\infty$, there exists a sequence $\{w_n\}_{n=1}^{\infty}$

such that $\lim_{n\to\infty} w_n = w$ and $w_n \in graphW_n$, for all n. Write $w_n = [W_n(\kappa_n)\ \kappa_n]'$, $w = [W_\infty(\kappa)\ \kappa]'$. By our definition of W_n, for every n there exists solutions $(g_n(\kappa_n)$, $b_n(\kappa_n)$, $G_n(\kappa_n)$, $B_n(\kappa_n))$ such that

$$\begin{aligned}\pi[u_1(g_n(\kappa_n)) + \delta W_{n-1}(G_n(\kappa_n))] + (1-\pi)[u_1(b_n(\kappa_n)) + \delta W_{n-1}(B_n(\kappa_n))] &= W_n(\kappa_n),\\ \pi[u_1(g_n(\kappa_n)) + \delta G_n(\kappa_n)] + (1-\pi)[u_1(b_n(\kappa_n)) + \delta B_n(\kappa_n)] &= \kappa_n,\end{aligned}$$

and as $\kappa_n \to \kappa$, the limits g, b, G, B of the sequences $\{g_n(\kappa_n)\}, \{b_n(\kappa_n)\}, \{G_n(\kappa_n)\}, \{B_n(\kappa_n)\}$ must exist and satisfy

$$\pi[u_1(g) + \delta G] + (1-\pi)[u_1(b) + \delta B] = \kappa.$$

Therefore $w = \lim_{n\to\infty} w_n =$

$$\begin{bmatrix} \pi u_2(1+e-g) + (1-\pi)u_2(1-b) + \delta \lim\limits_{n\to\infty}[\pi W_{n-1}(G_n(\kappa_n)) + (1-\pi)W_{n-1}(B_n(\kappa_n))] \\ \pi u_1(g) + (1-\pi)u_1(b) + \delta[\pi G + (1-\pi)B] \end{bmatrix}.$$

But as $\{W_n\}$ converges uniformly and $G_n(\kappa_n) \to G, B_n(\kappa_n) \to B$, we have

$$\lim_{n\to\infty}[\pi W_{n-1}(G_n(\kappa_n)) + (1-\pi)W_{n-1}(B_n(\kappa_n))] = \pi W_\infty(G) + (1-\pi)W_\infty(B).$$

Therefore

$$w = \begin{bmatrix} \pi u_2(1+e-g) + (1-\pi)u_2(1-b) + \delta[\pi W_\infty(G) + (1-\pi)W_\infty(B)] \\ \pi u_1(g) + (1-\pi)u_1(b) + \delta[\pi G + (1-\pi)B] \end{bmatrix},$$

and as $(g_n(\kappa_n)$, $b_n(\kappa_n)$, $G_n(\kappa_n)$, $B_n(\kappa_n))$ satisfy P, R, F' at all n, u_i are continuous and the constraint sets are closed, (g, b, G, B) must also satisfy P, R, F'. Therefore $w \in Q(graphW_\infty)$.■

9 References

Abreu, D., D. Pearce and E. Stacchetti (1990), "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring", *Econometrica 58* 1041-1063.

Altonji, J.G., F. Hayashi and L.J. Kotlikoff (1992), "Is the Extended Family Altruistically Linked? Direct Tests Using Micro Data", *American Economic Review 82*, 1177-1198.

Alvarez, F. and U.J. Jermann (2000), “Efficiency, Equilibrium, and Asset Pricing with Risk of Default”, *Econometrica 68,* 775-797.

Alvarez, F. and U.J. Jermann (2001), “Quantitative Asset Pricing Implications of Endogenous Solvency Constraints”, *Review of Financial Studies 14,* 1117-1151.

Atkeson, A. and R.E. Lucas, jr. (1992), “On Efficient Distribution with Private Information”, *Review of Economic Studies 59,* 427-453.

Cochrane, J.H. (1991), "A Simple Test of Consumption Insurance", *Journal of Political Economy 99*, 957-976.

Dixit, A., G.M. Grossman and F. Gul (2000), "The Dynamics of Political Compromise", *Journal of Political Economy 108*, 531-568.

Fafchamps, M. and S. Lund (2003), "Risk-sharing Networks in Rural Philippines", *Journal of Development Economics 71*, 261-287.

Flavin, M.A. (1981), "The Adjustment of Consumption to Changing Expectations About Future Income", *The Journal of Political Economy 89,* 974-1009.

Hall, R.E and F.S. Mishkin (1982), "The Sensitivity of Consumption to Transitory Income: Estimates from Panel Data on Households", *Econometrica 50*, 461-482.

Hamilton, B. H. (2000), "Does Entrepreneurship Pay? An Empirical Analysis of the Returns of Self-Employment", *Journal of Political Economy 108*, 604-631.

Kehoe, T.H. and D.K. Levine (1993), "Debt-Constrained Asset Markets", *Review of Economic Studies 60*, 865-888.

Kehoe, T.H. and D.K. Levine (2001), "Liquidity Constrained Markets versus Debt Constrained Markets", *Econometrica 69*, 575-598.

Kletzer, K.M. and B.D. Wright (2000), "Sovereign Debt as Intertemporal Barter", *American Economic Review 90,* 621-639.

Kocherlakota, N. (1996), “Implications of Efficient Risk-Sharing without Commitment”, *Review of Economic Studies 63,* 595-609.

Ligon, E. (1998), "Risk-Sharing and Information in Village Economies", *Review of Economic Studies 65*, 847-864.

Ligon, E., J. Thomas and T. Worrall (2002), "Informal Insurance Arrangements with Limited Commitment: Theory and Evidence from Village Economies", *Review of Economic Studies 69*, 209-244.

Moskowitz, T.J. and A. Vissing-Jorgensen (2002), "The Returns to Entrepreneurial Investment: A Private Equity Premium Puzzle?", *American Economic Review 92*, 745-778.

Nelson, J.A. (1994), "On Testing for Full Insurance using Consumer Expenditure Survey Data", *Journal of Political Economy 102*, 384-394.

Phelan, C. and R.M. Townsend (1991), “Computing Multi-Perid, Information-Constrained Optima”, *Review of Economic Studies 58,* 853-881.

Rockafellar, R. T.and R. J.-B.Wets (1998), "Variational Analysis", Berlin, Springer-Verlag.

Stokey, N.L. and R.E. Lucas, jr., with E.C. Prescott (1989), "Recursive Methods in Economic Dynamics", Cambridge, Harvard University Press.

Thomas, J. and T. Worrall (1988), “Self-Enforcing Wage Contracts”, *Review of Economic Studies 55,* 541-553.

Thomas, J. and T. Worrall (1990), “Income Fluctuation and Asymmetric Information: An Example of a Repeated Principal-Agent Problem”, *Journal of Economic Theory 51*, 367-390.

Townsend, R.M. (1994), "Risk and Insurance in Village India", *Econometrica 62,* 539-591.

Zeldes, S. P. (1989), "Consumption and Liquidity Constraints: An Empirical Investigation", *Journal of Political Economy 97*, 305-346.

Wang, C. (1995), "Dynamic Insurance with Private Information and Balanced Budgets", *Review of Economic Studies 62*, 577-595.