

Industrial dynamics, Schumpeter and the neoclassical growth model*

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Abstract

This paper studies industry level dynamics and demonstrates the ability of a modified neoclassical growth model to capture a range of empirical facts. The paper begins by using U.S. data to document skilled and unskilled labor trends within industry sector classifications as well as industry sector output trends. Using CPS data from 1968 to 2004, it is shown that the ratio of skilled workers to unskilled workers employed has risen in all industries. The absolute increase in this ratio was larger in the more skilled industries while the growth rate was larger in the less skilled industries. Furthermore, using national income account data it is shown that relatively high skilled industries have accounted for an increasing share of output over time. A version of the neoclassical growth model is then constructed to match these observations. One important feature of this model is a structure which introduces new goods into the economy at each moment of time. The model is able to capture a rich set of labor market movements between sectors and between skill levels as well as changes in the relative output shares across industries, yet preserves many nice features of the neoclassical growth model. This changing product mix, labor dynamics and changing industrial output composition can be interpreted with a Schumpeter viewpoint.

JEL Classification: E13; J20; O30

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1 Introduction

Economists have long pondered the impact of a changing industrial composition on the overall economy. Questions such as whether the decline in the manufacturing sector is a worrisome trend are only a recent variant of this question. Not long ago the discussion was more focused on particular parts of the manufacturing sector such as the textile industry.¹ Similarly, the prominence of recent advances in computing and information technologies has cast these industries as possible catalysts of improved economy-wide performance.² Despite the urgency to some of these inquiries, the foundation for these questions is quite old and is often traced to Schumpeter (1942) who described the evolution of the industrial sector as exhibiting a process of creative destruction through the introduction of new goods and technologies.

With all the changes occurring at the industry level, one might think the overall economy would also show signs of this churning. However, the aggregate goods market data trends have remained very stable over time. In fact, these aggregate trends are so well recognized and so consistent over just about any subinterval of data that they have become the cornerstone for the idea of balanced growth and the neoclassical growth model.³

Recent work has taken steps toward reconciling changes in the sectoral composition of output with this remarkable aggregate stability. Collectively, this work explores the modelling restrictions required for this reconciliation. Kongsamut, Rebelo, and Xie (2001) consider balanced growth in the face of a persistent reallocation of labor from agriculture to manufacturing and services. The key to aggregate stability in their model is a knife-edge relationship between a productivity parameter and a preference parameter. This relationship is employed also in Meckl (2002). Foellmi and Zweimüller (2002) build an endogenous growth model which also accommodates structural change in employment along a balanced growth path. In their model, dynamic differences in income elasticities across sectors give rise to sectoral dynamics where expanding and declining industries

¹The origin of this concern probably comes from local news outlets or trade publications that have more focused readerships. Headlines such as, “A house divided: Manufacturing in crises,” from the November 1st, 2005 issue of *Industry Week* or “Textile trade deficit hits all-time high,” from the March 7th, 2005 edition of the *Southwest Farm Press* indicate the feeling behind these changes. While academic studies such as Crandall (1993), Sachs and Shatz (1994) or Fisher and Rupert (2005), offer support to the decline in manufacturing, they offer more objective viewpoints about the costs and benefits of the changing industrial structure.

²For instance, Krueger (1993) has asked how computers have impacted wages and Greenwood and Yorukoglu (1997) have discussed the advantages new information technologies have for productivity.

³Kaldor is credited with bringing these facts to the attention of the profession through numerous reports in the 1950’s and 1960’s. Solow (1970), reflects on his work in growth and credits Kaldor (1961) as a source for the empirical facts. More recently these facts have inspired Lucas (1988) and Romer (1987) in their work on endogenous growth.

coexist. Balanced growth is achieved by assuming a particular willingness to substitute across goods in a hierarchical preferences structure. Most recently, Ngai and Pissarides (2006) build a model where structural change results from different rates of technological progress across industries and balanced growth arises because of a unit-elastic intertemporal elasticity of substitution.

This paper also considers trends in industrial composition in a structure that allows balanced growth. However, we focus not only on trends in sectoral employment but also in the skill composition of this employment. We begin by documenting labor and output trends in the thirteen major industrial classifications used by the U.S. Commerce Department. Labor trends are documented using U.S. Current Population Survey data from 1968 to 2004 while output trends are derived from U.S. national income account data from 1968 to 1999.

For each industry classification, we disaggregate employment to reveal the trends in unskilled and skilled labor employment. This disaggregation allows us to dichotomize industries as initially relatively high skilled or low skilled. We find that the ratio of skilled workers to unskilled workers has grown in each industry. The absolute increase in the ratio was largest in the initially skilled industries and the ratio has grown more in the unskilled industries. Furthermore, industries with a relatively high use of skilled workers have accounted for an increasing share of output over time.

With these empirical facts in hand, we build a neoclassical growth model consistent with these trends. The model shares some features with the one in Ngai and Pissarides (2006). In particular, good specific technological changes yield sectoral dynamics while unit intertemporal elasticity of substitution yields balanced growth. However, our model more closely resembles that of Blankenau and Cassou (2006) which also shares these features. In that paper, it is shown that the dynamics of time allocation across skilled and unskilled labor can be separated from the dynamics of aggregate output. This allows for balanced growth with a trend toward a more educated labor force.

This paper makes use of a similar structure but considers the case of several industries. In the version of the model used here, the initially high skilled industries have dynamic changes in their production processes that result in a more rapidly increasing need for high skilled workers. The equilibrium level of output also grows more in these industries and the growth rate in skilled workers is relatively smaller than in the initially low skilled industries.

We demonstrate that the changing industrial composition in the labor force is entirely consistent with our modified neoclassical growth model. We conclude that despite there being considerable turmoil at the industrial level, the aggregate economy can perform as in the standard neoclassical

growth model, with either transitional dynamics toward balanced growth or long sustained periods of balanced growth.

Beyond this, we uncover a richer set of dynamics in skilled labor trends and show that the neoclassical growth model requires only modest refinements to offer an explanation. This is an innovation in its own right but proves to be of greater importance. An implication of Ngai and Pissarides (2006) is that sectoral labor trends with balanced growth implies the validity of “Baumol’s cost disease” where an industry experiencing slow productivity growth consumes an ever-increasing share of labor. Our model suggests a different form of industry specific technological changes and implies an eventual stabilization of labor ratios.

The remainder of the paper is structured as follows. In section 2 the empirical facts are described. Section 3 presents the model and section 4 shows several general results implied by this model. In section 5 we provide several illustrations which simplify the model enough to see more clearly the presence of the dynamics we are interested in. Section 6 summarizes and concludes the paper.

2 The historical facts

There are four types of historical facts which this paper endeavors to model: (1) Stable aggregate ratios in the goods market; (2) An increasing fraction of the total labor force which is skilled; (3) Industry level labor dynamics which vary depending on whether the industry has relatively high skilled workers or relatively low skilled workers; And, (4) an increasing share of output is produced by the relatively high skilled industries. Two of these facts, stable aggregate ratios in the goods market over time and aggregate labor market trends, have been well documented elsewhere. The stable aggregate ratios in the goods market are so well known by the profession that they are often referred to as the Kaldor facts in tribute to Nicholas Kaldor who studied these ratios and brought them to the attention of the profession.⁴ The aggregate labor market trends, such as a rising use of skilled labor, were more recently recognized by the profession and have also drawn considerable attention.⁵

The labor market trends within the various industrial sectors of the economy are less well

⁴See, for instance, Kaldor (1961) for one of his presentations of the facts. These facts have been reviewed and the data series extended by Solow (1970), Romer (1987, 1990) and Blankenau and Cassou (2006).

⁵See, for instance, Denison (1985), Jorgenson, Gallop and Fraumeni (1987), Jones (2002) and Blankenau and Cassou (2006).

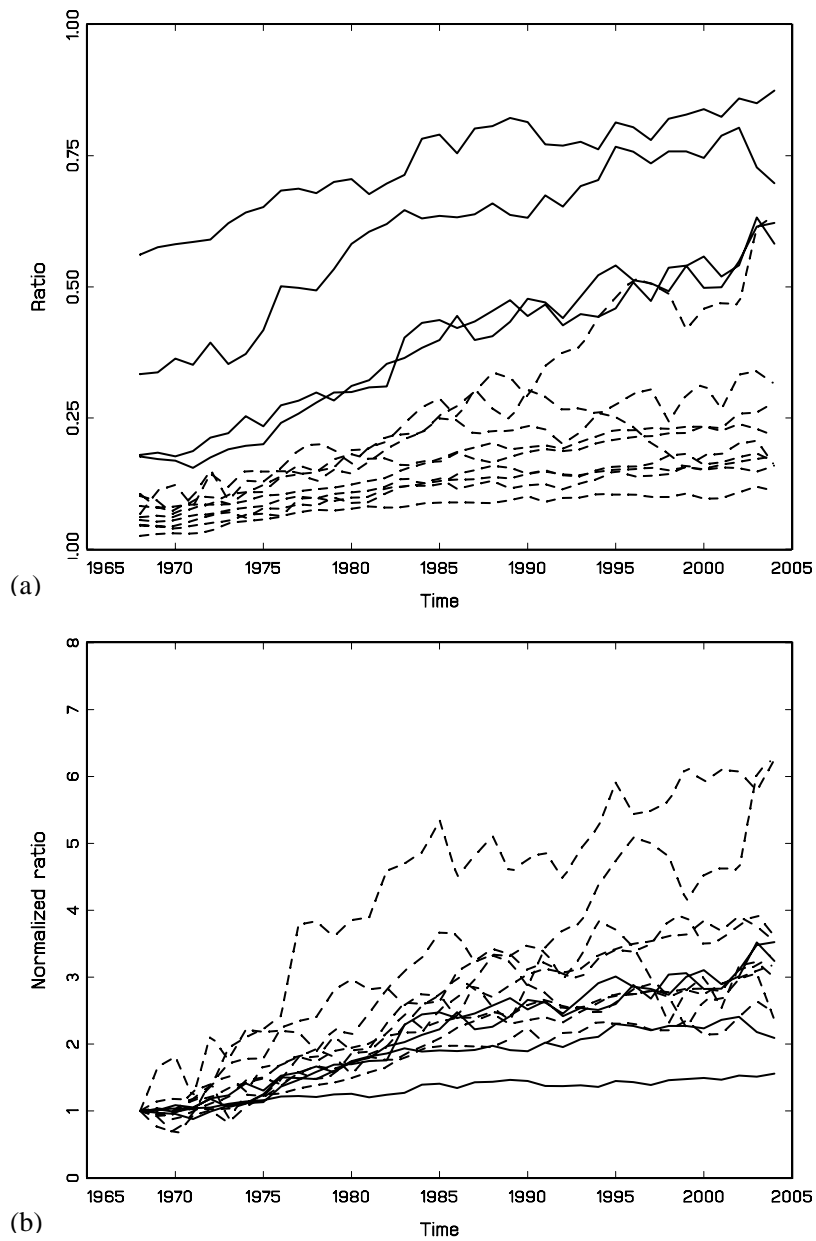


Figure 1: The ratio of skilled to unskilled labor by industry. Solid lines represent initial high skilled industries. Dashed lines represent initial low skilled industries. Panel (a) shows the ratios and panel (b) shows these ratios normalized by 1968 values.

known. Some of these trends are illustrated in Figure 1. Panel (a) shows the ratio of skilled labor to unskilled labor from 1968 to 2004 in thirteen broad industry classifications used by the U.S. Commerce Department.⁶ One readily apparent trend in this figure is that each sector of the economy increasingly uses skilled workers. Thus the aggregate trend toward increased skill levels holds at the industry level. However, the figure also illustrates that the trend toward skill is not uniform. Industries which were initially skilled generally have larger absolute increases in the ratio of skilled to unskilled workers. To make this more obvious, the initially high skilled industries are plotted with solid lines and the initially low skilled industries are plotted with dashed lines.

Another sectoral labor market trend is illustrated in panel (b), where the skilled to unskilled labor ratio is normalized by dividing by the initial value of the ratio. This can be used to compare how the ratio of skilled to unskilled labor has grown in the different industries. Panel (b) also makes use of the convention that initially high skilled industries are plotted with solid lines and initially unskilled industries are plotted with dashed lines. Flatter slopes for the initially high skilled industries indicates that the ratio has grown more slowly in these industries.

Table 1 further demonstrates these two sectoral trends. As the table shows, the initially high skilled industries tend to have larger increments in the ratio, but lower growth rates in the ratio. It is these two sectoral facts which we will pursue in our model. However, in our model, rather than keeping track of thirteen sectors, we simplify the analysis to just two. Figure 2 shows these trends for industry aggregates where the four initially high skilled industries are put together into one aggregate and the nine initially low skilled industries are put together into the second aggregate. As expected, the initially high skilled industry aggregate has a larger absolute increase in the ratio of skill to unskilled workers, but a lower growth rate of this ratio. Furthermore, the last two rows of Table 1 support this result by showing the values for the ratio and the normalized ratio at the beginning and end of the observation period.

⁶In the graph we do not identify the industries because the individual plots are hard to distinguish when so many lines are drawn with different line styles. However, in Table 1 below we do report information by industry. For further information on the data, see Appendix B.

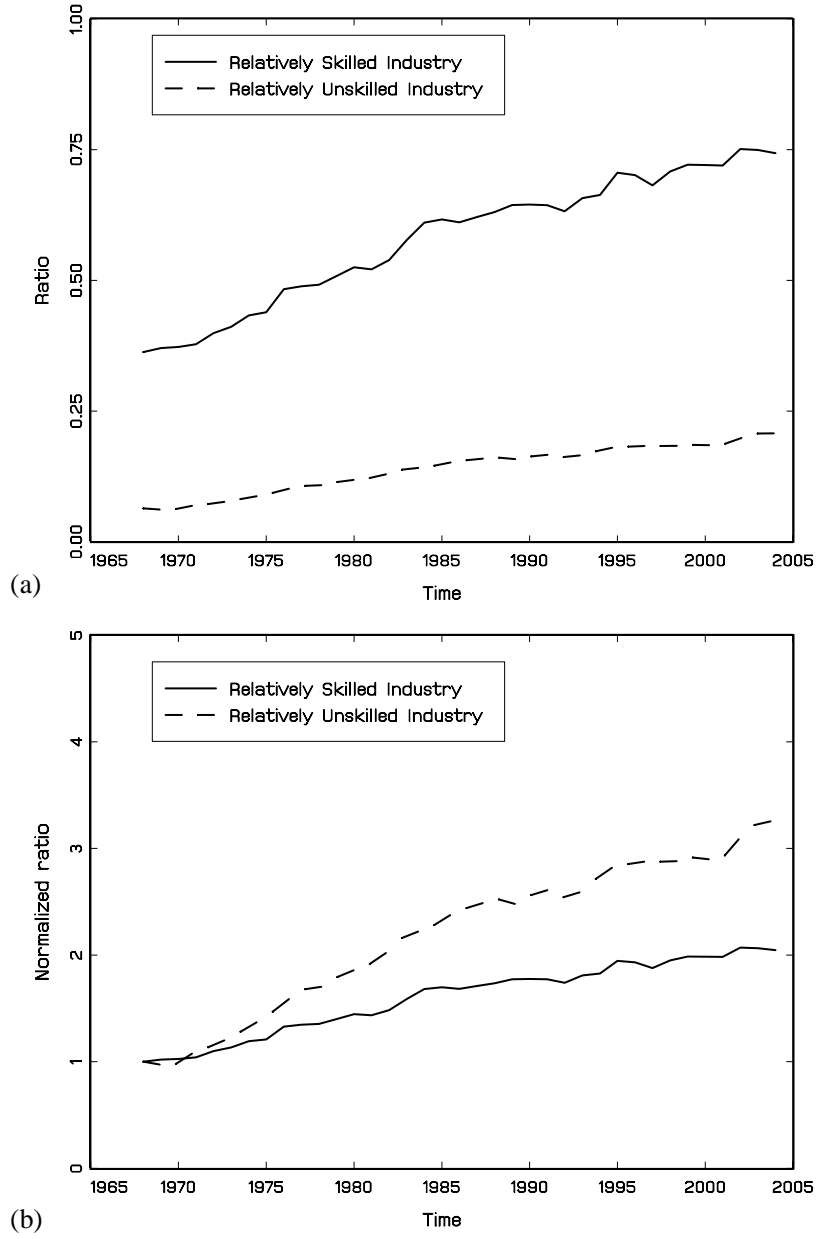


Figure 2: Ratio of skilled labor to unskilled labor: industry aggregates. Panel (a) shows the ratios and panel (b) shows these ratios normalized by 1968 values.

Table 1 - Skilled to unskilled labor ratios across industries

	Skill to unskill ratio			Normalized ratio		
	1968 Value	2004 Value	Change	1968 Value	2004 Value	Change
Initially skilled						
Educational and health services	.562	.874	.312	1.000	1.556	.556
Professional & business services	.334	.682	.348	1.000	2.042	1.042
Financial services	.180	.582	.403	1.000	3.242	2.242
Public administration	.177	.622	.445	1.000	3.523	2.523
Initially unskilled						
Leisure and hospitality service	.105	.487	.382	1.000	4.632	3.632
Information	.101	.636	.535	1.000	6.278	5.278
Manufacturing	.083	.279	.197	1.000	3.384	2.384
Mining	.068	.159	.092	1.000	2.355	1.355
Other services	.061	.216	.155	1.000	3.546	2.546
Wholesale & retail trade	.056	.177	.122	1.000	3.186	2.186
Transportation	.046	.167	.121	1.000	3.607	2.607
Construction	.045	.111	.066	1.000	2.473	1.473
Agriculture & forestry	.026	.159	.134	1.000	6.251	5.251
Skilled aggregate	.363	.741	.378	1.000	2.040	1.040
Unskilled aggregate	.064	.217	.153	1.000	3.404	2.404

The trend in output attributed to relatively high skilled industries and relatively low skilled industries is also less well known.⁷ In Figure 3 below, we plot the ratio of output between high skilled and low skilled industries using the categorization described in Table 1. This plot shows that over time the initially high skilled industries have accounted for an increasing share of overall GDP. This change appears to result from a long run trend.

3 The model

The model preserves most of the structure in Blankenau and Cassou (2006). A key distinction is to break the consumer good sector into two groups: initially high skilled and initially low skilled goods. This section begins by describing the corporate sector and the various production functions. Next the consumer sector is described and the competitive equilibrium concept defined. As in Blankenau and Cassou (2006), the model is formulated in intensive form with exogenous growth.⁸

⁷Some popular press single industry anecdotes, however, are well known. For instance, the fact that the manufacturing industry has accounted for a declining share of GDP is widely reported and discussed in mainstream media.

⁸An appendix describing an aggregate form of the model and its conversion into the intensive form can be obtained from the authors upon request.

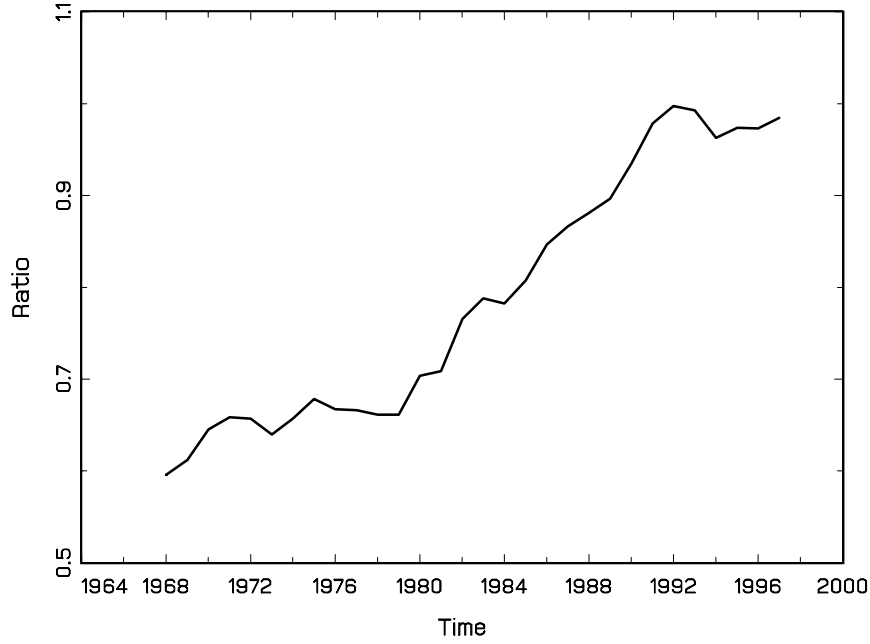


Figure 3: Ratio of output in the initially skilled sector to output in the initially unskilled sector.

3.1 The corporate sector

There are three types of producers in the corporate sector. One type produces an investment good and the other two produce consumer goods. The assumption that capital goods are built in a separate sector is used in part to deflect concern that the results are connected to any assumption regarding where capital goods are produced. Consumer goods fall into two categories based on the importance of skilled labor in their production as made explicit below. All investment and consumption goods are produced from capital, skilled labor and unskilled labor. Production technologies for the investment good and all consumer goods exhibit constant returns to scale. In this case, we lose no generality in assuming that a single firm produces each type of good.

First consider the firm producing the investment good. We use the notation ι to identify variables associated with this sector. The firm produces according to

$$\begin{aligned}
 y_{\iota,t} &= k_{\iota,t}^{\alpha} \left[\gamma_{\iota} s_{\iota,t}^{\sigma} + (1 - \gamma_{\iota}) u_{\iota,t}^{\sigma} \right]^{\frac{1-\alpha}{\sigma}} \quad \text{for } \sigma \neq 0, \quad \text{and} \\
 y_{\iota,t} &= k_{\iota,t}^{\alpha} \left[s_{\iota,t}^{\gamma_{\iota}} u_{\iota,t}^{1-\gamma_{\iota}} \right]^{1-\alpha} \quad \text{for } \sigma = 0,
 \end{aligned} \tag{1}$$

where $0 \leq \gamma_\iota \leq 1$, $0 \leq \alpha \leq 1$ and $\sigma \leq 1$. The labor aggregate in equation (1), given by $[\gamma_\iota s_{\iota,t}^\sigma + (1 - \gamma_\iota) u_{\iota,t}^\sigma]^{\frac{1-\alpha}{\sigma}}$, is a constant elasticity of substitution combination of skilled labor, $s_{\iota,t}^\sigma$ and unskilled labor, $u_{\iota,t}^\sigma$. The parameter σ determines the elasticity of substitution (equal to $\frac{1}{1-\sigma}$) and γ_ι determines the relative importance of each in determining the size of the aggregate. This aggregate combines with capital, $k_{\iota,t}$, to produce units of the investment good, $y_{\iota,t}$. Given the Cobb-Douglas specification, α is the elasticity of output with respect to the capital input and $(1 - \alpha)$ is the elasticity of output with respect to the labor aggregate.

The consumer goods sector consists of two sectors. One industry is initially more skill intensive. We refer to this as the initially skilled sector and denote it by a . We refer to the other as the initially unskilled sector and denote it by b . It is essential for our purposes that the relative skill intensity of production increase in each sector. This could be accomplished with a single good in each industry where the production technology of each good undergoes a skill biased technological change. However, such an approach implies price behavior that is not realistic.⁹ Instead, we model each sector more generally as containing many goods with an expanding product space. Our specification allows the possibility of changes in the production technologies of existing products or the introduction of new goods with new production technologies.

The expanding product space fits the Schumpeter notion of creative destruction as new goods and new technologies reduce the importance of existing goods. This leads to a more natural interpretation of technological change and allows a varied set of possible dynamics. In our specification new goods in each industry have a higher need for skill than the average prior goods in that industry. In addition, new goods that enter industry a have a higher need for skilled labor than the new goods that enter industry b thus making the skill demand for industry a rise faster than that in industry b . Collectively, these features cause the average skill intensity of each sector to rise while the trend is more pronounced in the initially skilled industry.

The number of goods (and firms) at time t in sector j is denoted by $n_{j,t}$, and grows according to $n_{j,t} = n_{j,0} e^{g_j t}$ where $g_j \geq 0$. Our results are not tied to the relative rates of growth for the two sectors and to emphasize this we set the growth rates equal unless otherwise specified. The

⁹See Blankenau and Cassou (2006) for more detail on this.

production function for a representative firm is given by

$$\begin{aligned}
y_{j,\omega,t} &= k_{j,\omega,t}^\alpha \left[\gamma_{j,\omega,t} s_{j,\omega,t}^\sigma + (1 - \gamma_{j,\omega,t}) u_{j,\omega,t}^\sigma \right]^{\frac{1-\alpha}{\sigma}} \text{ for } \sigma \neq 0, \text{ and} \\
y_{j,\omega,t} &= k_{j,\omega,t}^\alpha \left[s_{j,\omega,t}^{\gamma_{j,\omega,t}} u_{j,\omega,t}^{(1-\gamma_{j,\omega,t})} \right]^{1-\alpha} \text{ for } \sigma = 0,
\end{aligned} \tag{2}$$

where $0 \leq \gamma_{j,\omega,t} \leq 1$ and $\omega \in [0, n_{j,t}]$. The j notation indicates the category of good, a or b , and the ω notation identifies a particular good in that category. Hence $y_{j,\omega,t}$ is the output of good ω of industry j at time t , $k_{j,\omega,t}$ is the amount of capital employed in its production and $s_{j,\omega,t}$ and $u_{j,\omega,t}$ are the skilled and unskilled labor inputs. The parameters α and σ play the same roles as in the investment good sector. Note that we allow the γ parameters in the goods sector to differ across time, goods, and category. This will prove pivotal in generating the dynamics of interest. In contrast γ_ι is fixed. A constant γ parameter for the numeraire good also proves important in reconciling these dynamics with balanced growth.

The richness of the dynamic structure requires some compromise of precision to keep the notation succinct. First, since $n_{a,t}$ may not equal $n_{b,t}$, the support of $\omega \in [0, n_{j,t}]$ may differ by industry. It would be more precise then, for example, to write $y_{j,\omega(j),t}$ indicating the sector specificity of ω . However, no later confusion arises with our abbreviated notation of $y_{j,\omega,t}$. Furthermore, it will turn out that any item indexed by $j \in \{a, b\}$ is in general time specific even along the balanced growth path. In this sense, the time notation is redundant for these items. To further simplify notation, we hereafter drop the time subscript for all items indexed by $j \in \{a, b\}$. With this further simplified notation, $y_{j,\omega}$ is the output of good ω in industry j at time t and other industry specific items are similarly defined.

Equations (1) and (2) are clearly generalizations of the Cobb-Douglas production function prevalent in the growth literature. In our model, as in the simpler Cobb-Douglas case, labor receives a constant share of output given by $(1 - \alpha)$. We follow the growth literature in justifying its use by noting the lack of trend in this share in the U.S. and other economies. In contrast, the share of this labor income accruing to skilled and unskilled labor is not constant. We can capture a trend in labor shares across education levels with trends in $\gamma_{j,\omega}$ even in the case where $\sigma = 0$. However, we opt to consider the more general case where $\sigma \neq 0$ for several reasons. First, we want to emphasize that our results are not contingent upon unit elasticity of skilled and unskilled labor as the Cobb-Douglas case might suggest. Secondly, we want to discuss the importance of this elasticity in determining our results. Aside from these expositional considerations, we note that

empirical observations suggest that the true elasticity is in fact greater than 1.¹⁰

We assume that factors of production are freely mobile. This implies that factor prices are equal across firms and along with the first order conditions for firm optimal hiring choices give,

$$r_t = r_{\iota,t} = r_{j,\omega} = \frac{\partial y_{\iota,t}}{\partial k_{\iota,t}} = p_{j,\omega} \frac{\partial y_{j,\omega}}{\partial k_{j,\omega}}, \quad (3)$$

$$w_t^s = w_{\iota,t}^s = w_{j,\omega}^s = \frac{\partial y_{\iota,t}}{\partial s_{\iota,t}} = p_{j,\omega} \frac{\partial y_{j,\omega}}{\partial s_{j,\omega}}, \quad (4)$$

$$w_t^u = w_{\iota,t}^u = w_{j,\omega}^u = \frac{\partial y_{\iota,t}}{\partial u_{\iota,t}} = p_{j,\omega} \frac{\partial y_{j,\omega}}{\partial u_{j,\omega}}. \quad (5)$$

where r_t is the rental rate of capital, w_t^s is the wage to a unit of skilled labor and w_t^u is the wage to a unit of unskilled labor. The price of the investment good is $p_{\iota,t}$. Our numeraire good is capital. Since a unit of the investment good is the same as an ex-dividend unit of capital, this requires normalizing $p_{\iota,t} = 1$. Given this, other prices, $p_{j,\omega}$, are stated in terms of the investment or capital good.

3.2 The consumer sector

The economy is populated by an infinitely-lived representative household with lifetime utility defined over an index of current consumption, c_t ,

$$\int_0^{\infty} e^{-\rho t} \ln(c_t) dt, \quad (6)$$

where $\rho > 0$. The consumption index is given by

$$c_t \equiv \left[\frac{1}{n_a + n_b} \left[\sum_{j=a,b} \int_0^{n_j} x_{j,\omega}^{1-\psi} d\omega \right] \right]^{\frac{1}{1-\psi}} \quad \text{for } \psi \neq 1, \text{ and} \quad (7)$$

$$c_t \equiv \exp \left[\frac{1}{n_a + n_b} \left[\sum_{j=a,b} \int_0^{n_j} \ln x_{j,\omega} d\omega \right] \right] \quad \text{for } \psi = 1,$$

where $x_{j,\omega}$ indicates the demand for good ω in industry j at date t . The parameter $\psi > 0$ is the intratemporal elasticity of substitution across goods so the second case in (7) arises in the case of Cobb-Douglas or unit elastic preferences.

The consumer faces a goods constraint at each date. Let k_t be the total capital stock per effective labor unit at date t (hereafter ‘capital’), and $0 < \delta < 1$ be the rate at which this capital

¹⁰See Katz and Murphy (1992) and Blankenau (1999).

depreciates. Furthermore, let g_A and g_L be the exogenous rates of technological progress and population growth. Then the goods constraint is given by

$$\begin{aligned} \dot{k}_t &= r_t k_{l,t} + w_t^s s_{l,t} + w_t^u u_{l,t} \\ &+ \sum_{j=a,b} \int_0^{n_j} [r_t k_{j,\omega} + w_t^s s_{j,\omega} + w_t^u u_{j,\omega}] d\omega \\ &- \sum_{j=a,b} \int_0^{n_j} x_{j,\omega} p_{j,\omega} d\omega - (\delta + g_A + g_L) k_t. \end{aligned} \quad (8)$$

Adding the terms to the right of the equality in the first line to the integral in the second line gives total payments to factors of production; i.e. total income. The integral in the third line gives total consumption spending. Income less consumption spending is investment which we denote by i_t . Thus equation (8) reduces to the familiar law of motion for the capital stock given by $\dot{k}_t = i_t - (\delta + g_A + g_L) k_t$.

To arrive at a measure of total output, we follow the convention of weighting the quantities of each good by their market prices. Period t output, then, is

$$y_t = y_{l,t} + \sum_{j=a,b} \int_0^{n_j} y_{j,\omega} p_{j,\omega} d\omega. \quad (9)$$

Similarly, the total capital stock is

$$k_t = k_{l,t} + \sum_{j=a,b} \int_0^{n_j} k_{j,\omega} d\omega. \quad (10)$$

Since the price of capital is 1 regardless of where it is employed, the capital stock is not weighted by prices.

The agent also faces a time constraint. In each period the agent has an endowment of one unit of time. Since leisure is not valued, the agent allocates time to maximize labor income net of the education cost of acquiring skill. The cost of education could be modelled as a time cost, a goods cost, or a combination of both. Including both costs proves redundant and we prefer to follow Lucas (1988) and many others in modelling a time cost. This is consistent with the upward trend in college enrollment and duration seen throughout most of the post World War II years in the United States. Milesi-Ferretti and Roubini (1998) show that the nature of the education cost is important in analyzing tax policy. However, we are not considering issues related to taxation and the distinction is not as important.

The essential feature of time allocation is that it is costly to refine the time endowment for the provision of skilled labor. So long as this cost exists, its specific nature or timing proves less

important. Thus we model the education requirement for skill as linear and contemporaneous. Specifically, in order to provide a unit of skilled labor, $\frac{1}{\theta}$ units of time must be spent in education. Given this, the total amount of skill provided in the labor market, S_t , is related to time spent in education, e_t , according to

$$S_t \equiv s_{l,t} + \sum_{j=a,b} \int_0^{n_j} s_{j,\omega} d\omega = \theta e_t, \quad (11)$$

and the time constraint is

$$u_{l,t} + \sum_{j=a,b} \int_0^{n_j} u_{j,\omega} d\omega + s_{l,t} + \sum_{j=a,b} \int_0^{n_j} s_{j,\omega} d\omega + e_t = 1. \quad (12)$$

With this, the model is fully specified and we are able to define an equilibrium in this economy.

Definition: A competitive equilibrium is a set of infinite price sequences

$\{r_{l,t}, w_{l,t}^s, w_{l,t}^u, [[p_{j,\omega}, r_{j,\omega}, w_{j,\omega}^s, w_{j,\omega}^u, 0 \leq \omega \leq n_j] : j = a, b] : 0 \leq t\}$ and infinite allocation sequences $\{k_t, y_{l,t}, i_t, k_{l,t}, s_{l,t}, u_{l,t}, [[y_{j,\omega}, x_{j,\omega}, k_{j,\omega}, s_{j,\omega}, u_{j,\omega}, 0 \leq \omega \leq n_j] : j = a, b], e_t : 0 \leq t\}$ with k_0 given such that:

1. Given prices, firms maximize profits subject to production constraints (1) and (2). With factor mobility this yields (3), (4), and (5).
2. Given prices, consumers maximize utility (6) subject to resource constraints (8), (11) and (12).
3. Markets clear:
 - (a) Capital goods produced equals investment: $y_{l,t} = i_t$.
 - (b) Consumption goods produced equals consumption goods demand: $y_{j,\omega} = x_{j,\omega}$ for $0 \leq \omega \leq n_{jt}$ and $j = a, b$.
 - (c) Capital input demand equals capital input supplied: (10).
 - (d) Labor input demand equals labor input supplied: (11) and (12).¹¹

4 Implications of the general model

In this section we present several general results and organize them into three subsections. The first subsection shows that the separation result from Blankenau and Cassou (2006) generalizes

¹¹Equations (11) and (12) appear as both constraints to the consumer and labor market clearance conditions because we have assumed there to be a single representative agent.

into the modeling structure described here.¹² This separation result shows that the dynamics of aggregate output can be tracked without knowledge of the sectoral composition of this output or the dynamics of time allocations. This result is useful because it means that we can concentrate on sectoral dynamics and labor market dynamics knowing that any trends in those parts of the economy will be consistent with balanced growth in total output.

The second and third subsections present results on industrial labor market dynamics and industrial output composition respectively. These subsections explain both intuitively and formally how the model can achieve these dynamics. Later, in section 5, we provide several sample economies to illustrate these results.

4.1 Separation Theorem

To describe the equilibrium dynamics, it proves convenient to introduce ν_t as a measure of the share of the capital stock used to produce investment goods. It will be shown shortly that ν_t also represents the time used in the production of investment goods.

We also will make extensive use of the following z variables which are defined for each good and are related to each good's γ value. Although the γ values are the fundamental distinguishing characteristic for each good and account for all the dynamics presented here, there are many occasions where thinking about things in terms of the z values proves to be advantageous. For instance, it will be shown later that these z terms will provide a nice formulation for relative prices and they will also have a useful interpretation as labor productivity parameters. With that in mind, we define

$$z_t \equiv \left[\gamma_t^{\frac{1}{1-\sigma}} \left(\frac{\theta}{1+\theta} \right)^{\frac{\sigma}{1-\sigma}} + (1 - \gamma_t)^{\frac{1}{1-\sigma}} \right]^{\frac{1}{\sigma}},$$

$$z_{j,\omega} \equiv \left[\gamma_{j,\omega}^{\frac{1}{1-\sigma}} \left(\frac{\theta}{1+\theta} \right)^{\frac{\sigma}{1-\sigma}} + (1 - \gamma_{j,\omega})^{\frac{1}{1-\sigma}} \right]^{\frac{1}{\sigma}},$$

and an aggregation of these by

$$Z_j \equiv \int_0^{n_j} z_{j,\omega}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)}{\psi}} d\omega. \quad (13)$$

Recall that all items with subscript $j \in \{a, b\}$ are time specific. Thus $z_{j,\omega}$ differs across both goods and time while z_t is constant.

¹²Since this result is a generalization of the Blankenau and Cassou (2006) result, it is presented here with no proof. Readers interested in greater intuition and formal details can consult Blankenau and Cassou (2006). Readers interested in the formal proof can obtain one by contacting the authors.

Proposition 1. Household Allocations.

(a) Capital and labor allocations: There exists a ν_t such that capital is allocated according to

$$k_{i,t} = \nu_t k_t, \quad (14)$$

$$k_{j,\omega} = k_t (1 - \nu_t) z_{j,\omega}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)}{\psi}} (Z_a + Z_b)^{-1}, \quad (15)$$

and time is allocated according to

$$u_{i,t} = \nu_t (1 - \gamma_i)^{\frac{1}{1-\sigma}} z_i^{-\sigma}, \quad (16)$$

$$s_{i,t} = \nu_t \left[\gamma_i \frac{\theta}{1 + \theta} \right]^{\frac{1}{1-\sigma}} z_i^{-\sigma}, \quad (17)$$

$$u_{j,\omega} = [1 - \gamma_{j,\omega}]^{\frac{1}{1-\sigma}} (1 - \nu_t) z_{j,\omega}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} (Z_a + Z_b)^{-1}, \quad (18)$$

$$s_{j,\omega} = \left[\gamma_{j,\omega} \frac{\theta}{1 + \theta} \right]^{\frac{1}{1-\sigma}} (1 - \nu_t) z_{j,\omega}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} (Z_a + Z_b)^{-1}. \quad (19)$$

(b) Dynamics and convergence: The dynamics of ν_t and k_t are governed by¹³

$$\frac{\dot{\nu}_t}{1 - \nu_t} = \alpha k_t^{\alpha-1} z_i^{(1-\sigma)(1-\alpha)} (1 - \nu_t) - (1 - \alpha) (\delta + g_A + g_L) - \rho, \quad (20)$$

$$\frac{\dot{k}_t}{k_t} = z_i^{(1-\sigma)(1-\alpha)} \nu_t k_t^{\alpha-1} - (\delta + g_A + g_L), \quad (21)$$

where equations (20) and (21) describe a globally stable system converging to a path with $\dot{\nu}_t = \dot{k}_t = 0$.

(c) Output: The total value of output is given by

$$y_t = k_t^\alpha z_i^{(1-\sigma)(1-\alpha)}. \quad (22)$$

Part (a) of the proposition shows how labor and capital are allocated for production of the investment good and the various consumption goods. These equations show that at each point in time, the same share of capital and time, ν_t , is allocated to provide the investment good and the complement is allocated to providing consumption goods. That ν_t represents the share of capital allocated to the investment good is obvious from equation (14). To see that ν_t is also the share of time allocated to the investment good, recall that $\frac{1}{\theta}$ units of time must be spent in education to

¹³Because $\frac{\dot{\nu}_t}{1 - \nu_t} = \frac{1 - \dot{\nu}_t}{1 - \nu_t}$, $\frac{\dot{\nu}_t}{1 - \nu_t}$ can be interpreted as the growth rate of $1 - \nu_t$.

provide a unit of skilled labor. Because of this, $s_{\iota,t} \left(1 + \frac{1}{\theta}\right)$ units of time are required to provide $s_{\iota,t}$ units of skilled labor in the investment sector so that the total time allocated to this sector is $u_{\iota,t} + s_{\iota,t} \left(1 + \frac{1}{\theta}\right)$. From equations (16) and (17) and the definition of z_{ι} ,

$$u_{\iota,t} + s_{\iota,t} \left(1 + \frac{1}{\theta}\right) = \nu_t. \quad (23)$$

Since only 1 unit of time is available, ν_t is the share devoted to investment and $1 - \nu_t$ is the share allocated to consumption goods.

Part (b) shows that given some initial level of ν and k , the dynamics of both can be tracked without any knowledge of the consumption sector. To see this, simply note the absence of any consumption market indicator in equations (20) and (21). Thus the separation holds both along the transition path and in balanced growth. However, as the U.S. experience is broadly consistent with the construct of a balanced growth path, we are particularly interested in this case. A second implication of part (b) of the proposition is that such a balanced growth path does exist since ν_t and k_t converge to steady state levels ν and k . At this point, a constant share of all resources are devoted to the production of the investment good.

Part (c) of the proposition shows that output can be measured without knowledge of any indicators from the consumption goods sector. Again, this is clear from the absence of such indicators in equation (22). With ν and k constant along the balanced growth path, y is also constant so that aggregate output grows at rate $g_A + g_L$ as in a standard Ramsey-Cass-Koopmans growth model.

For our purposes, there are two key implications of parts (b) and (c). First, the dynamics of output can be tracked without knowledge of how time is allocated within the consumption goods sectors. Equation (23) helps to explain this feature of the model. In equilibrium, the relative value of output in the investment good or consumption good categories reflects the value of the inputs used in the respective categories. Since ν_t and $1 - \nu_t$ are the shares of both labor and capital allocated to producing the investment good and consumption goods, $\frac{\nu_t}{1 - \nu_t}$ is the value of the investment good relative to the consumption good. Thus knowing the value of investment is sufficient to find the value of output. Note that to find the value of any particular consumption good, one must know how the $1 - \nu_t$ units of time and the $(1 - \nu_t) k_t$ units of capital are allocated across the various goods. However, this allocation does not influence the total value created in the consumption sector.

The second key implication is that ν_t eventually converges to a steady state level. Since dynamics

persist in the consumption sector, this implies that the allocation of resources across investment and consumption is independent of such dynamics. Equation (9) shows that if output and investment have converged to a steady state, so has the value of consumption goods. However, this is an aggregation of prices and quantities and does not require that prices of consumption goods have stabilized. In fact, the dynamics of prices prove closely related to our results in the next section. In order to have balanced growth, we need to neutralize the effect of price changes in the consumption goods sector on the savings rate as expressed by ν . The unit elastic intertemporal rate of substitution inherent with logarithmic preferences assures the independence of price dynamics and savings.

While we are not able to relax the unit elastic intertemporal rate of substitution assumption and preserve balanced growth, we are able to defend it as a reasonable approximation for our purposes. As mentioned before, Kongsamut, Rebelo, and Xie (2001) and Meckl (2002) assume a particular relationship between a technology parameter and a preference parameter. In addition, Blankenau and Cassou (2006) and Ngai and Pissarides (2006) also make the same assumption held in this paper. There are several advantages to this choice. First, at the aggregate level the resulting framework is precisely the Ramsey, Cass, Koopmans model and thus both well understood and widely accepted. Secondly, the value is empirically defensible. While a wide range of estimates populate the literature, a value close to 1 for the intertemporal elasticity of substitution is not uncommon.¹⁴ Finally, while deviations from 1 disallow balanced growth, small deviations from 1 are likely to lead to small deviations from balanced growth. A balanced growth path would likely be a reasonable approximation to observed dynamics over modest intervals.

Finally, before turning to the labor market and output dynamics, it is useful to elaborate on one of the more intuitive interpretations of $z_{j,\omega}$. Using equations (2), (14), (18) and (19), it can be shown that

$$\frac{y_{j,\omega}}{u_{j,\omega} + s_{j,\omega} \left(1 + \frac{1}{\theta}\right)} = k_t^\alpha z_{j,\omega}^{(1-\sigma)(1-\alpha)}. \quad (24)$$

The left-hand side is the equilibrium output per unit of time (i.e. labor productivity) for good (j, ω) . To see this, note that the denominator is the equilibrium amount of time used in the production of good (j, ω) inclusive of education costs. Since k_t is constant in balanced growth, $z_{j,\omega}^{(1-\sigma)(1-\alpha)}$ scales this equilibrium productivity measure and we can think of $z_{j,\omega}$ as a determinant of equilibrium labor productivity.

¹⁴See for example Beaudry and van Wincoop (1996), Vissing-Jørgensen (2002), Zhang (2006) and the calibration exercises of Prescott (1985) and Jones, Manuelli and Siu (2000).

4.2 Industrial labor market dynamics

In this subsection we explain how the model is capable of reproducing the empirical fact of industry specific labor dynamics even along a balanced growth path. The principle insight for this result comes from equations (18) and (19). These demonstrate that even with ν_t constant in balanced growth, the allocation of time to each good changes with $\gamma_{j,\omega}$ or Z_j . To show that industry level labor dynamics persist with balanced growth, we need to demonstrate that the labor aggregates within each industry change through time. Since our data is in terms of the ratio of skilled to unskilled labor, we focus on this aggregate. It is straight forward to show that

$$\frac{S_j}{U_j} \equiv \frac{\int_0^{n_j} u_{j,\omega} d\omega}{\int_0^{n_j} s_{j,\omega} d\omega} = \frac{\int_0^{n_j} [1 - \gamma_{j,\omega}]^{\frac{1}{1-\sigma}} z_{j,\omega}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} d\omega}{\int_0^{n_j} \left[\gamma_{j,\omega} \frac{\theta}{1+\theta} \right]^{\frac{1}{1-\sigma}} z_{j,\omega}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} d\omega}. \quad (25)$$

To interpret how this ratio evolves over time, note that changes arise from two sources. First, the ratio can change over time because new goods in the industry have values for γ_{j,n_j} that move the ratio and second it can change if any existing good, $\gamma_{j,\omega} \in [\gamma_{j,0}, \gamma_{j,n_j})$, has a change in its $\gamma_{j,\omega}$ value. Furthermore, since $\frac{S_j}{U_j}$ does not depend on any economic variables and is entirely determined by the $\gamma_{j,\omega}$ parameters, the key to generating the observed industry level dynamics is for the time paths of $\gamma_{j,\omega} \in [\gamma_{j,0}, \gamma_{j,n_j}]$ to be such that the firm level equations (18) and (19) aggregate in equation (25) to the proper industry level dynamics. In section 5 below, we show that such industry level dynamics are easy to achieve by constructing several examples which provide both analytical and numerical demonstrations of the sought after dynamics.

4.3 Industrial output composition

The model is also capable of reproducing a changing industrial output composition even when aggregate growth is balanced. To intuitively understand this possibility, consider a simple situation in which the number of goods in each sector is equal at each date. In particular, each sector starts with the same number of goods and the growth rates for goods in each sector are the same. This means that at each instant of time, new goods appear in pairs, with one new good in each sector. Next focus on the demand curves for these new goods. Since the elasticity of substitution in the utility function is the same for each good, the demand curves for each good will be the same. Consider the situation where $\psi < 1$.¹⁵ This implies that demand functions for all goods

¹⁵The discussion for $\psi > 1$ is similar while the case in which $\psi = 1$ results in the industrial shares that are unchanging over time. The result when $\psi = 1$ is discussed more fully in example 2 below.

are relatively elastic. So to produce the desired changes in the output shares, we need for the marginal goods in sector a to generate relatively greater output values than the marginal goods in sector b . This will arise so long as the supply curve for the marginal good in sector a is further to the right than the supply curve for the marginal good in sector b . But this can be assured through appropriate relative values for the marginal good's $\gamma_{j,\omega}$. In section 5 below, several specific formulations for $\gamma_{j,\omega}$ processes that achieve this result are provided.

To more formally understand this possibility, it is necessary to first define sectoral output. As is common, we measure each good's contribution to total output by its market value. To get such an expression, begin by substituting (15), (18) and (19) into (2) and make use of the definition of $z_{j,\omega}$ to get

$$y_{j,\omega} = \frac{z_{j,\omega}^{\frac{(1-\alpha)(1-\sigma)}{\psi}} (1 - \nu_t) k_t^\alpha}{Z_a + Z_b}. \quad (26)$$

An analogous derivation can obtain an expression for $y_{l,t}$, which can then be substituted along with (26) into (3) to obtain

$$p_{j,\omega} = \frac{z_l^{(1-\sigma)(1-\alpha)}}{z_{j,\omega}^{(1-\sigma)(1-\alpha)}}. \quad (27)$$

These equations imply that

$$y_j = \int_0^{n_j} y_{j,\omega} p_{j,\omega} d\omega = \frac{(1 - \nu_t) k_t^\alpha z_l^{(1-\sigma)(1-\alpha)}}{Z_a + Z_b} \int_0^{n_j} \frac{z_{j,\omega}^{\frac{(1-\alpha)(1-\sigma)(1-\psi)}{\psi}}}{z_{j,\omega}^{\frac{(1-\alpha)(1-\sigma)(1-\psi)}{\psi}}} d\omega. \quad (28)$$

and along with equation (13) yield

$$\frac{y_a}{y_b} = \frac{Z_a}{Z_b}. \quad (29)$$

From this and equation (24) we conclude that in equilibrium, the relative size of industries and the labor productivities in those industries are closely related. This is also a feature in Ngai and Pissarides (2006). It is an artifact of perfect factor mobility and technology induced structural change and is likely to arise in other models that share these features.

It is clear from equation (29) that many kinds of behavior for this output ratio are possible. We are most interested in situations where industry a becomes a larger share of total output over time. We emphasize that all that is needed to generate an upward trend in industry a 's share of output is for Z_a to grow more rapidly than Z_b . This ultimately depends upon the component $z_{j,\omega}$ values.

To intuitively trace these dynamics again it is useful to consider industries with equal numbers of goods and equal rates of good introduction so that goods are introduced in pairs, one for each

industry. Begin by focusing on the case in which goods are relatively substitutable; $\psi < 1$. In this case, goods with higher values of $z_{j,\omega}$ have higher labor productivity and have greater resources employed in the goods production.¹⁶ With both more resources and higher productivity, the output of goods with higher $z_{j,\omega}$ values is higher. Of course the equilibrium price is lower but because demand is elastic on net there is a higher equilibrium value. Because both marginal goods have the same demand equations, the industry which experiences the largest increase in value will be the one which has the marginal good that has the largest productivity. To summarize, in the substitutable good case, the sector which experiences a rising share of output has new goods with relatively greater productivities.

Next focus on the case in which goods are relatively complementary: $\psi > 1$. In this case, goods with higher values of $z_{j,\omega}$ have higher productivity but also have fewer resources employed in the goods production. This lower resource input in part offsets the higher productivity. The net effect is a higher output (see equation (26)) but along with the lower price, this is enough to imply the value of the output is lower because demand is inelastic. Because both marginal goods have the same demand equations, the industry which adds the smallest increment to its value will be the one which has the marginal good that has the largest productivity. In other words, in the complementary good case, the sector which experiences a rising share of output actually has new goods with relatively lower productivities. This surprising result follows because complementarity in utility implies a preference for keeping consumption levels more equal across goods. Thus the marginal good which has a high productivity will not see as large a difference in equilibrium production as in the substitutable good case because resources are shifted to other production activities in order to maintain this more equal consumption preference.

5 Illustrations

There are two sets of sectoral dynamics that we would like our model to exhibit along a balanced growth path. First, the sector experiencing more rapid skill biased technological change should experience larger increases but slower growth in the ratio of skilled to unskilled labor employed and second, this sector should account for an increasing share of output. Section 4 makes clear that these dynamics can be achieved when $\gamma_{j,\omega}$ follows appropriate time paths. In this section we provide several specific formulations which demonstrate these dynamics more clearly or provide

¹⁶It can be shown that the share of time and capital used for good (j, ω) , is $(1 - \nu_t) z_{j,\omega}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)}{\psi}} (Z_a + Z_b)^{-1}$.

further insight into what is necessary to achieve them.

The first example simplifies the economy considerably and is designed primarily to shed light on Baumol's cost disease which suggests that an industry experiencing slow productivity growth consumes an ever-increasing share of income. It is shown that this disease is not necessarily present in our set up and distinguishes our results from Ngai and Pissarides (2006) where the disease is present.

The second example is designed to provide more insight into what is necessary to achieve the two dynamic trends. It works with a simple unit elastic utility function and Cobb-Douglas production function and shows that the integrals in (25) can be easily evaluated. Then with both a general and specific formulation for $\gamma_{j,\omega}$, it is shown how the desired labor dynamics can be achieved. This example then goes on to show that the unit elastic utility function will not be able to achieve the desired industrial share dynamics except when sector a has a higher rate of new good introduction than sector b . Although this may seem like a negative result, it is actually positive because it shows that to obtain both types of dynamic results when each sector experiences an equal rate of new good introduction, one must move away from the intratemporal unit elastic utility function. The third example simply generalizes this second example slightly to achieve both dynamic results. This third example also has a nice Schumpeter like interpretation.

5.1 Output dynamics and Baumol's cost disease

To explore this issue, it is enough to work with a simple model where only sector a experiences technological change and goods within each sector have identical technologies. In this case, we can use equations (18) and (19) to show the ratio of skilled labor to unskilled labor in sector a is

$$\frac{S_a}{U_a} = \left[\frac{\gamma_a \frac{\theta}{1+\theta}}{1 - \gamma_a} \right]^{\frac{1}{1-\sigma}}. \quad (30)$$

Note that the ω notation has been dropped since goods are identical. From this, it is clear that $\dot{\gamma}_a > 0$ is sufficient to assure that this ratio is increasing through time. Furthermore, equation (29) in this case reduces to

$$\frac{y_a}{y_b} = \frac{z_a \frac{(1-\alpha)(1-\psi)(1-\sigma)}{\psi}}{Z_b} = \left[\gamma_a^{\frac{1}{1-\sigma}} \left(\frac{\theta}{1+\theta} \right)^{\frac{\sigma}{1-\sigma}} + (1 - \gamma_a)^{\frac{1}{1-\sigma}} \right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)}{\sigma\psi}} Z_b^{-1}.$$

In discussing the dynamics, we consider only the empirically relevant case where $\sigma > 0$.¹⁷ This is for brevity and a symmetric set of result exists with $\sigma < 0$. Proposition 2 shows conditions under

¹⁷Katz and Murphy (1992) estimate σ near .29 and Blankenau (1999) finds $\sigma=.414$.

which the desired dynamics arise:

Proposition 2. Suppose goods within each sector have identical production technologies, $\dot{\gamma}_a > 0$, $\dot{\gamma}_b = 0$ and $\sigma > 0$. Then the ratio of skilled workers to unskilled workers grows in industry a and is fixed in industry b . Furthermore, the relative share of output in industry a grows if $\psi < 1$ and $\gamma_a > \frac{1+\theta}{1+2\theta}$ or if $\psi > 1$ and $\gamma_a < \frac{1+\theta}{1+2\theta}$.

There are two keys to understanding this result. The first involves the relationship between z_a and the value of total output. If $\psi < 1$, goods are relative substitutes and an increase in z_a increases the value of output as discussed above. The second involves the relationship between γ_a and z_a . This is a non-monotonic relationship. When γ_a is relatively large (small), z_a is increasing (decreasing) in γ_a . Thus we conclude that when γ_a is sufficiently large, further increases will increase z_a and with $\psi < 1$ this increases the relative value of its output. Alternately, when γ_a is sufficiently small, increases in γ_a decrease z_a and with $\psi > 1$ this increases the relative value of its output.

The conditions of Proposition 2 make it easy to generate a dynamic economy where the industry experiencing skill biased technological change accounts for a growing share of total output. Considering, for example, the following path for γ_a assuming $\psi < 1$:

$$\gamma_a = \underline{\gamma}_a + (\bar{\gamma}_a - \underline{\gamma}_a) \left(\frac{t}{1+t} \right),$$

where $\underline{\gamma}_a \geq \frac{1+\theta}{1+2\theta}$ is constant and $\bar{\gamma}_a = 1$. Then in each period the skill ratio grows in industry a and its share of output grows. The ratio of output in the two industries however converges to

$$\frac{y_a}{y_b} = \left(\frac{\theta}{1+\theta} \right)^{\frac{(1-\alpha)(1-\psi)}{\psi}} Z_b^{-1}.$$

This result indicates that in the limit one industry does not dominate the other. Furthermore, it is straightforward to show that in the limit

$$L_a = \frac{(1-\nu) \left(\frac{\theta}{1+\theta} \right)^{\frac{(1-\alpha)(1-\psi)}{\psi}}}{\left(\frac{\theta}{1+\theta} \right)^{\frac{(1-\alpha)(1-\psi)}{\psi}} + Z_b}.$$

Since L_a is the share of time devoted to industry a and its limiting value is less than 1, we see that one sector does not end up consuming all resources. One interpretation of this is that Baumol's cost disease need not be an implication of observed sectoral shifts.

5.2 Implications when $\psi = 1$ and $\sigma = 0$

Here we focus on conditions necessary to achieve the kind of industrial sector labor market dynamics seen in section 2. Because the integrals in (25) are not possible to solve generally we consider the

case where intratemporal utility is logarithmic ($\psi = 1$) and production is Cobb-Douglas ($\sigma = 0$). By setting $\psi = 1$ and $\sigma = 0$, (18) and (19) simplify and can be integrated across sectors to produce

$$U_j = \int_0^{n_j} u_{j,\omega} d\omega = (1 - \nu_t) (1 - \bar{\gamma}_j) \frac{n_j}{n_a + n_b} \quad \text{and} \quad (31)$$

$$S_j = \int_0^{n_j} s_{j,\omega} d\omega = \frac{\theta}{1 + \theta} (1 - \nu_t) \bar{\gamma}_j \frac{n_j}{n_a + n_b}, \quad (32)$$

where $\bar{\gamma}_j = \frac{1}{n_j} \int_0^{n_j} \gamma_{j,\omega} d\omega$; i.e $\bar{\gamma}_j$ is the average skill share in sector j .

For ease of notation it proves convenient to define $\tilde{\gamma}_j \equiv \frac{\bar{\gamma}_j}{1 - \bar{\gamma}_j}$ for $j = a, b$. This is increasing in $\bar{\gamma}_j$ and thus rises as the relative importance of skill in industry j rises. For this reason we refer to $\tilde{\gamma}_j$ as the skill share ratio. The following proposition relates changes in the skill share ratio to changes in the skill ratio.

Proposition 3. Suppose $\psi = 1$ and $\sigma = 0$. If at each date t ,

$$1 > \frac{\dot{\tilde{\gamma}}_a}{\tilde{\gamma}_a} > \frac{\dot{\tilde{\gamma}}_b}{\tilde{\gamma}_b}, \quad (33)$$

then at each date t , movement in skill shares are related according to

$$\left(\frac{\dot{S}_a}{U_a} \right) > \left(\frac{\dot{S}_b}{U_b} \right) \quad \text{and} \quad (34)$$

$$\frac{\left(\frac{\dot{S}_b}{U_b} \right)}{\left(\frac{S_b}{U_b} \right)} > \frac{\left(\frac{\dot{S}_a}{U_a} \right)}{\left(\frac{S_a}{U_a} \right)}. \quad (35)$$

The left-hand side of the condition expressed in equation (33) requires that the skill share ratio for the initially low skilled industry grows more rapidly than for the initially high skilled industry. The right-hand side indicates that the disparity in growth rates cannot be too large. This is because we need to allow for larger absolute increases in the skill ratio for the skilled industry. If these conditions are met, the behavior of skill shares anticipated by the neoclassical growth model are precisely those observed in the data.

Since $\tilde{\gamma}_b < \tilde{\gamma}_a$ by definition, there is clearly some ratio of growth rates that will satisfy this condition at any time, t . The right-hand ratio in (33) is clearly growing when the middle condition is satisfied. Thus (33) will be satisfied in all time periods only if the middle ratio is increasing more rapidly than the right-hand ratio and is also converging to something less than or equal to one.

The observed labor market dynamics, then will arise given any process for $\gamma_{j,\omega}$ that satisfies condition (33). In the following section we demonstrate that if one is willing to rely on numerical

solutions, it is not difficult to arrive at processes for $\gamma_{j,\omega}$ which meet the requirements. Before turning to this though, we demonstrate that even analytical results are available. We need a process for $\gamma_{j,\omega}$ for which $\frac{1}{n_j} \int_0^{n_j} \gamma_{j,\omega} d\omega$ is tractable and can be shown to satisfy equation (33) at each moment. One candidate is the vintage goods case discussed in Blankenau and Cassou (2006). Under this formulation, the production elasticity for skilled labor for good ω in sector j at time t is given by

$$\gamma_{j,\omega} = \underline{\gamma}_j + (\overline{\gamma}_j - \underline{\gamma}_j) \left(\frac{(\phi + 1)\omega^\phi + \omega^{2\phi}}{(1 + \omega^\phi)^2} \right), \quad (36)$$

where $0 < \phi \leq 1$, and $\overline{\gamma}_j > \underline{\gamma}_j$ with both constant. Notice that $\underline{\gamma}_j$ gives the lower limit for the sector j skilled labor elasticity and $\overline{\gamma}_j$ gives the upper limit. In this formulation the elasticity for any product (j, ω) does not change over time and is only a function of the product's type. Also note that $\frac{\partial \gamma_{j,\omega}}{\partial \omega} > 0$, so that at any point in time, newer goods are more skill intensive.

We assume that at each date the two sectors have the same number of goods. Since we wish to construct the example with industry a as the higher skilled industry, we assume $\underline{\gamma}_a > \underline{\gamma}_b$. The relationship between $(\overline{\gamma}_a - \underline{\gamma}_a)$ and $(\overline{\gamma}_b - \underline{\gamma}_b)$ is less important and for simplicity it is easiest to just assume they are equal. Together these imply $\overline{\gamma}_a > \overline{\gamma}_b$. We also need to make sure that the growth rates for labor inputs within each sector have the proper relationships which requires that n_0 is large and that $.5 > \overline{\gamma}_a > \overline{\gamma}_b$. These assumptions are sufficient to establish the following result.

Proposition 4. Suppose $\psi = 1$ and $\sigma = 0$, $\gamma_{j,\omega}$ is described by (36), that n_0 is sufficiently large, $.5 > \overline{\gamma}_a > \overline{\gamma}_b$ and $(\overline{\gamma}_a - \underline{\gamma}_a) = (\overline{\gamma}_b - \underline{\gamma}_b)$. Then labor market dynamics match those in equations (34) and (35).

One downside of the assumptions in this subsection is that they imply that the only way for the ratio of output between the relatively high skilled and relatively low skilled sectors can increase over time is for sector a to have a higher rate of new good creation. This can be seen by noting that when $\psi = 1$, equation (29) reduces to

$$\frac{y_a}{y_b} = \frac{n_a}{n_b}.$$

Thus one industry eventually becomes inconsequential. In addition, it can be shown that in the limit all resources used in the production of consumption goods are consumed by the growing industry. By relaxing the assumptions that $\psi = 1$ and $\sigma = 0$, in the following section we are able to explain observed dynamics even with the relative product space in each industry constant and avoid these implications.

5.3 Complete dynamics and a Schumpeter interpretation

The previous subsection showed how the model can match the labor market facts, but the implications for the industrial output shares was lacking. In this section we show that by relaxing the assumptions that $\psi = 1$ and $\sigma = 0$, we can obtain a model with many goods in which sector a not only becomes increasingly skilled, but also produces an increasing share of total output even when new good creation is equal in the two sectors.

At this level of generality for the model, analytical results are not available. So to make headway we simulate the model numerically. To implement this we set

$$\gamma_{j,\omega} = \underline{\gamma}_j + (\overline{\gamma}_j - \underline{\gamma}_j) \left(\frac{\omega}{1 + \omega} \right)^\phi \left(\frac{\ln \omega}{\ln n_j} \right)^\lambda \quad (37)$$

where $0 < \underline{\gamma}_j < \overline{\gamma}_j < 1, 0 \leq \phi, 0 \leq \lambda$. This is a parsimonious yet general specification for the $\gamma_{j,\omega}$ process. To understand the implications of this functional form, first focus on the case where $\phi > 0, \lambda = 0$. With these restrictions, any good, once introduced has the same production technology forever. However, new goods (with a higher ω) are more skill intensive. Thus the introduction of new goods increases the average skill share in each industry. Next, consider the case where $\lambda > 0, \phi = 0$. Here each new good in industry j has $\gamma_{j,\omega} = \overline{\gamma}_j$. To see this note that when a good is introduced, it is the frontier good and $n_j = \omega$. As time passes and the frontier grows, the skill share of each existing goods falls. This reflects the possibility that as a good has been in production longer, simplification of the production process allows the producer to substitute lower cost unskilled labor for skilled labor. Aside from seeming to be a natural process, it is one supported by empirical evidence.¹⁸ With this as the only source of dynamics, there would be a gradual decrease in the need for and employment of skilled labor. However, this is countered by a growing product space. As the share of the product space with relatively high skilled labor needs grows, the equilibrium share of the workforce with skill grows with it. In the general case with $\lambda, \phi > 0$ the two sources of dynamics work in tandem generating a rich set of dynamics where the frontier goods are increasing in skill content and existing goods are going through a process of “simplifying by doing” yet aggregate output is growing at a steady pace.

To demonstrate that this functional form yields results consistent with the empirical facts we need to choose parameter values for the model. Aside from σ, θ , and α we have little guidance in how

¹⁸See Adler and Clark (1991) for example.

to choose parameters.¹⁹ Fortunately, little guidance is needed since our goal is only to demonstrate the possible simultaneity of balanced growth and industrial dynamics. For this purpose we employ the following set of parameters:

Parameter	σ	α	ψ	θ	$\bar{\gamma}_a$	$\bar{\gamma}_b$	$\underline{\gamma}_a$	$\underline{\gamma}_b$	$n_{j,0}$	$\frac{n_j}{n_j}$	λ	ϕ
Value	.4	.4	.5	10	.6	.4	.3	.1	1	.02	1	1

Since the model is no longer analytically tractable, we simulate the model numerically. The results of this exercise are summarized in Figure 4.

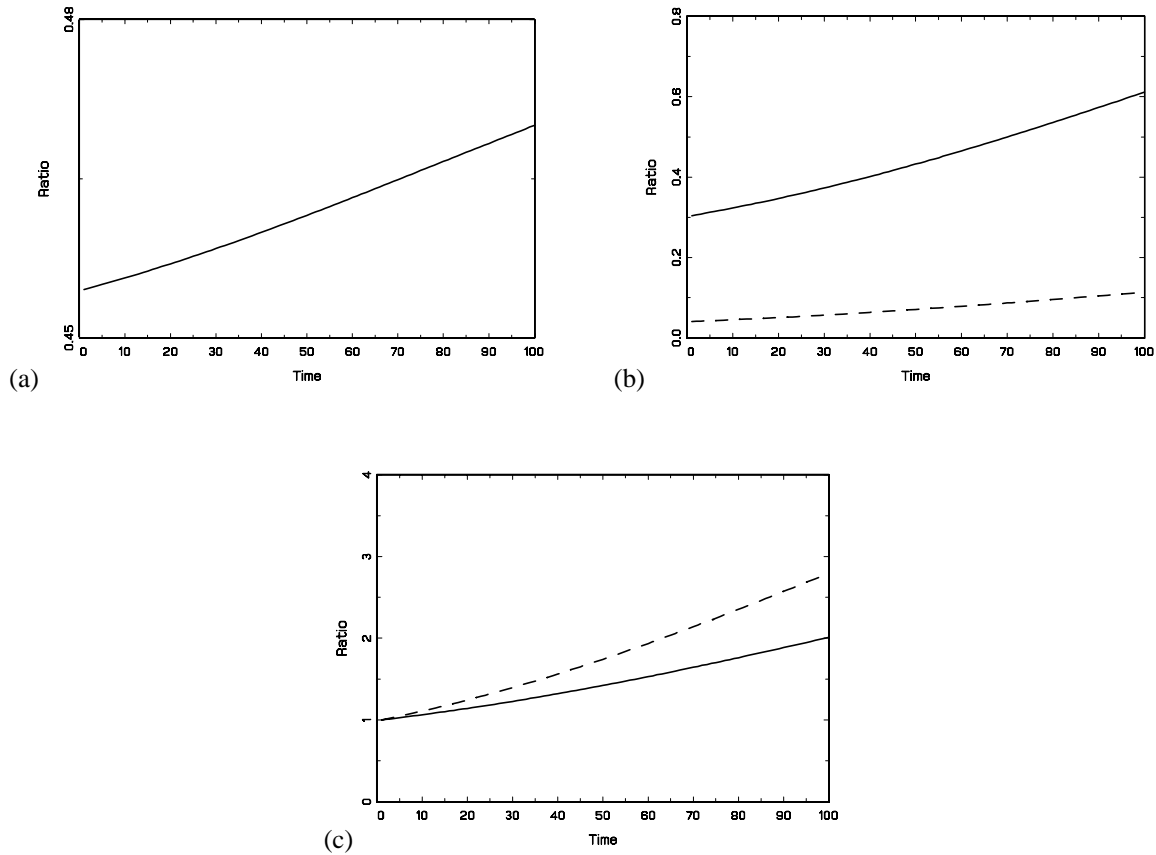


Figure 4: Panel (a) shows the ratio of output in industry *a* to that in industry *b* for the specification above. Panel (b) shows the ratio of skilled to unskilled labor employed in industry *a* (solid line) and industry *b* (dashed line). Panel (c) shows these ratios normalized by initial levels. The solid line again indicates industry *a*.

¹⁹See Katz and Murphy (1992) and Blankenau (1999).for estimates of σ . Researchers often set α in the range .3 to .4. A value of 10 for θ is consistent with 4 years of college for a 40 year career.

Panel (a) demonstrates that this model captures the desired industrial share dynamics. It shows that the value of output in industry a , grows more rapidly than industry b and thereby accounts for an increased share of total output. Panels (b) and (c) demonstrate that the model captures the labor trend dynamics. In panel (b) we see that both industries experience increases in the employment of skilled labor relative to unskilled labor and that the absolute increase in this ratio is largest in industry a . Panel (c) then computes this ratio normalized by its initial level. With this normalization it is clear that the ratio grows more rapidly in industry b .

These dynamics are similar to the observed dynamics in both industrial output and labor market trends which were describe in section 2. While the results are particular to our specification of the process governing $\gamma_{j,\omega}$, we note that many plausible processes could yield similar results. For example, setting either ψ or ϕ equal to 0 and recalibrating other parameters yields pictures very similar to those above. If we remove the \ln operator in equation (37) results are again similar so long as $\phi > 0$.²⁰ In fact, we find that the lessons learned in the simpler cases can provide guidance for what is needed to produce the observed dynamics in the more complicated cases. If we set the initial value of $\gamma_{j,\omega}$ sufficiently large, $\sigma > 0$, $\psi < 1$ and specify an appropriate process for $\gamma_{j,\omega}$, the results can be recreated. Furthermore, with $\sigma < 0$ and the initial value of $\gamma_{j,\omega}$ sufficiently small, similar dynamics arise.

Finally it is interesting to note that this example fits the Schumpeter notion of creative destruction because it has new goods entering the economy at each instant of time. These new goods that enter industry a have a higher need for skilled labor than the new goods that enter industry b thus making the skill demand for industry a rise faster than that in industry b . One could interpret these new goods which need increased skilled labor inputs as being those creative goods that are replacing the older goods. Under this interpretation, one might think of industry a as having greater improvements from its new goods which not only destroy (reduce in importance) the older industry a goods, but also destroy (reduce in importance) everything in industry b .

6 Summary and Conclusion

This paper shows that it is possible to have a host of industrial sector dynamics within a structure that aggregates up to the standard neoclassical growth model. This demonstration is important because data shows that some industrial sectors gain in size in the overall economy and others

²⁰In this case simplifying by doing is too rapid and the skilled ratio falls without setting $\phi > 0$ to counter this.

decline in size in the overall economy. It is sometimes speculated that industries in decline point to an ailing economy. What the demonstration here shows is that it is possible to have these dynamic industrial changes occurring yet the overall economy remains balanced and healthy.

We achieve these results by modifying a special case of the Ramsey, Cass, Koopmans model of exogenous growth. Other than our assumption of unit elastic intertemporal elasticity of substitution, our model at the aggregate level is a simple restatement of this venerable workhorse of growth theory. At the industry level, however, it is a rich generalization of the one and two sector growth models often built on this framework. The richness of this generalization is made possible by a theorem indicating that any sort of sectoral output and labor dynamics can be made consistent with balanced growth in aggregate.

This freedom in modeling industrial dynamics is restricted by empirical observations. We uncover several features of industrial dynamics that the model should encompass. Some of these, we feel, have not been well documented in prior literature. In particular we show that the skill intensive industries have been growing more rapidly and have experienced larger absolute increases, but slower growth rates, in the ratio of skilled to unskilled labor. Guided by these observations, we specify processes of technological change that recreate these results in a competitive equilibrium.

To understand the workings of our model more fully, we specify some simple versions of the model where the intuition is apparent before turning to numerical results for the full model. In the full model, the dynamics are indeed quite rich. We specify a process whereby the skill content of new goods grows through time while goods once introduced can become more or less skill intensive as learning occurs. All the while, observed industrial dynamics persist along a balanced growth path.

We note that our separation theorem can be taken as supportive of much earlier work in growth theory which ignores salient trends that seem to loom large for the macroeconomy. The trend toward decreased agricultural output and then manufacturing as shares of output are historical examples. The explosion of computing and information technologies and products may serve as current examples. However, we see the separation as having perhaps a more meaningful implication. Our results indicate that researchers interested in industrial sector trends may benefit from conducting analysis within the context of general equilibrium models that preserve the stylized fact of balanced growth. By digging deeper into industrial dynamics and considering the equilibrium consequences of such dynamics, new insights arise. In the example provided here, the dynam-

ics required to reproduce the empirical observations lead to interesting long run implications for industrial dynamics. We show that a continuation of the technological change consistent with recent experience is an eventual leveling out of the industrial composition of output and the skill composition of labor.

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A Appendix A: Proofs

A.1 Proof of Proposition 2

The first statement is clear upon taking the time derivative of equation (30). To verify the second statement, note that

$$\frac{\dot{y}_a}{y_b} = \frac{(1-\alpha)(1-\psi)}{\sigma\psi} \frac{y_a}{z_a^\sigma} \left[\gamma_a^{\frac{\sigma}{1-\sigma}} \left(\frac{\theta}{1+\theta} \right)^{\frac{\sigma}{1-\sigma}} - (1-\gamma_a)^{\frac{\sigma}{1-\sigma}} \right] Z_b^{-1} \dot{\gamma}_a.$$

With $\sigma > 0$, $\psi < 1$, this is positive so long as the bracketed expression is positive. This requires $\gamma_a > \frac{1+\theta}{1+2\theta}$. With $\sigma > 0$, $\psi > 1$, this is positive so long as the bracketed expression is negative. This requires $\gamma_a < \frac{1+\theta}{1+2\theta}$.

A.2 Proof of Proposition 3

Before starting the proof, it will be useful to find expressions for the time derivative of the skilled to unskilled labor ratio and the growth rate of the skilled to unskilled labor ratio. To this end, note that for any industry j , (31) and (32) give

$$\frac{S_j}{U_j} = \left(\frac{\theta}{1+\theta} \right) \frac{\bar{\gamma}_j}{(1-\bar{\gamma}_j)}. \quad (38)$$

so that

$$\left(\frac{\dot{S}_j}{\dot{U}_j} \right) = \left(\frac{\theta}{1+\theta} \right) \left(\frac{1}{(1-\bar{\gamma}_j)^2} \right) \dot{\bar{\gamma}}_j. \quad (39)$$

Using (38) and (39) gives

$$\frac{\left(\frac{\dot{S}_j}{\dot{U}_j} \right)}{\frac{S_j}{U_j}} = \frac{1}{\bar{\gamma}_j(1-\bar{\gamma}_j)} \dot{\bar{\gamma}}_j. \quad (40)$$

We are now ready to prove the theorem. First note that

$$\dot{\bar{\gamma}}_j = \frac{\dot{\bar{\gamma}}_j(1-\bar{\gamma}_j) + \bar{\gamma}_j\dot{\bar{\gamma}}_j}{(1-\bar{\gamma}_j)^2} = \frac{\dot{\bar{\gamma}}_j}{(1-\bar{\gamma}_j)^2}. \quad (41)$$

Next, note that the right side inequality in (33) gives

$$\dot{\bar{\gamma}}_a > \dot{\bar{\gamma}}_b,$$

which upon substitution of (41) gives

$$\frac{1}{(1-\bar{\gamma}_a)^2} \dot{\bar{\gamma}}_a > \frac{1}{(1-\bar{\gamma}_b)^2} \dot{\bar{\gamma}}_b.$$

Combining this with (39) gives (34). Second, note that the left side inequality in (33) gives

$$\frac{\dot{\bar{\gamma}}_b}{\bar{\gamma}_b} > \frac{\dot{\bar{\gamma}}_a}{\bar{\gamma}_a},$$

which upon substitution of $\tilde{\gamma}_j \equiv \frac{\bar{\gamma}_j}{1-\bar{\gamma}_j}$ and (41) gives

$$\frac{\dot{\bar{\gamma}}_b}{\bar{\gamma}_b(1-\bar{\gamma}_b)} > \frac{\dot{\bar{\gamma}}_a}{\bar{\gamma}_a(1-\bar{\gamma}_a)}.$$

Combining this with (40) gives (35).

A.3 Proof of Proposition 4

First note that

$$\int_0^{n_j} \gamma_{j,\omega} d\omega = \left(\underline{\gamma}_j + (\bar{\gamma}_j - \underline{\gamma}_j) \frac{\omega^{\phi+1}}{1+\omega^\phi} \right) \Big|_0^{n_j}.$$

Differentiate to verify. This gives

$$\bar{\gamma}_j = \frac{1}{n_j} \int_0^{n_j} \gamma_{j,\omega} d\omega = \underline{\gamma}_j + (\bar{\gamma}_j - \underline{\gamma}_j) \frac{n_j^\phi}{1+n_j^\phi}$$

so that

$$\tilde{\gamma}_j = \frac{\bar{\gamma}_j}{1-\bar{\gamma}_j} = \frac{\underline{\gamma}_j + (\bar{\gamma}_j - \underline{\gamma}_j) \frac{n_j^\phi}{1+n_j^\phi}}{1 - \underline{\gamma}_j - (\bar{\gamma}_j - \underline{\gamma}_j) \frac{n_j^\phi}{1+n_j^\phi}} = \frac{\underline{\gamma}_j + \bar{\gamma}_j n_j^\phi}{(1-\underline{\gamma}_j) + (1-\bar{\gamma}_j)n_j^\phi}. \quad (42)$$

Next note that

$$\dot{\tilde{\gamma}}_j = (\bar{\gamma}_j - \underline{\gamma}_j) \frac{\phi n_j^{\phi-1}}{(1+n_j^\phi)^2} \dot{n}_j,$$

so that

$$\frac{\dot{\tilde{\gamma}}_j}{(1-\bar{\gamma}_j)^2} = \frac{(\bar{\gamma}_j - \underline{\gamma}_j) \frac{\phi n_j^{\phi-1}}{(1+n_j^\phi)^2} \dot{n}_j}{\left(1 - \underline{\gamma}_j - (\bar{\gamma}_j - \underline{\gamma}_j) \frac{n_j^\phi}{1+n_j^\phi}\right)^2} = \frac{(\bar{\gamma}_j - \underline{\gamma}_j) \phi n_j^{\phi-1} \dot{n}_j}{\left((1-\underline{\gamma}_j) + (1-\bar{\gamma}_j)n_j^\phi\right)^2}. \quad (43)$$

Since this derivative is positive it shows that the average skill ratio is always increasing in each sector. Next we need to note a few relationships for $\underline{\gamma}_j$ and $\bar{\gamma}_j$ terms. It is straightforward to show the following hold: $\underline{\gamma}_a > \underline{\gamma}_b$, $1 - \underline{\gamma}_a < 1 - \underline{\gamma}_b$, $(1 - \bar{\gamma}_a)n_j^\phi < (1 - \bar{\gamma}_b)n_j^\phi$, $(1 - \underline{\gamma}_a) + (1 - \bar{\gamma}_a)n_j^\phi < (1 - \underline{\gamma}_b) + (1 - \bar{\gamma}_b)n_j^\phi$ and $\frac{1}{(1-\underline{\gamma}_a)+(1-\bar{\gamma}_a)n_j^\phi} > \frac{1}{(1-\underline{\gamma}_b)+(1-\bar{\gamma}_b)n_j^\phi}$.

We need to show first that (42) implies that at each date t , $\tilde{\gamma}_a > \tilde{\gamma}_b$. But this follows since $\underline{\gamma}_a + \bar{\gamma}_a n_j^\phi > \underline{\gamma}_b + \bar{\gamma}_b n_j^\phi$ and $\frac{1}{(1-\underline{\gamma}_a)+(1-\bar{\gamma}_a)n_j^\phi} > \frac{1}{(1-\underline{\gamma}_b)+(1-\bar{\gamma}_b)n_j^\phi}$.

Now are now ready to verify (33). Consider the inequality on the right side of (33) first. Plugging (43) into (41), we must show that

$$\frac{\frac{(\bar{\gamma}_a - \underline{\gamma}_a) \phi n_j^{\phi-1} \dot{n}_j}{\left((1-\underline{\gamma}_a)+(1-\bar{\gamma}_a)n_j^\phi\right)^2}}{\frac{(\bar{\gamma}_b - \underline{\gamma}_b) \phi n_j^{\phi-1} \dot{n}_j}{\left((1-\underline{\gamma}_b)+(1-\bar{\gamma}_b)n_j^\phi\right)^2}} = \frac{\frac{(\bar{\gamma}_a - \underline{\gamma}_a)}{\left((1-\underline{\gamma}_a)+(1-\bar{\gamma}_a)n_j^\phi\right)^2}}{\frac{(\bar{\gamma}_b - \underline{\gamma}_b)}{\left((1-\underline{\gamma}_b)+(1-\bar{\gamma}_b)n_j^\phi\right)^2}} > 1.$$

But this follows because $(\overline{\gamma_a} - \underline{\gamma_a}) = (\overline{\gamma_b} - \underline{\gamma_b})$ and $\frac{1}{(1-\underline{\gamma_a})+(1-\overline{\gamma_a})n_j^\phi} > \frac{1}{(1-\underline{\gamma_b})+(1-\overline{\gamma_b})n_j^\phi}$. Next focus on the left inequality in (33). From (41), (42), and (43), we need to show

$$1 > \frac{\frac{\tilde{\gamma}_a}{\overline{\gamma_a}}}{\frac{\tilde{\gamma}_b}{\overline{\gamma_b}}} = \frac{\frac{(\overline{\gamma_a} - \underline{\gamma_a})\phi n_j^{\phi-1}}{((1-\underline{\gamma_a})+(1-\overline{\gamma_a})n_j^\phi)(\underline{\gamma_a} + \overline{\gamma_a}n_j^\phi)}}{\frac{(\overline{\gamma_b} - \underline{\gamma_b})\phi n_j^{\phi-1}}{((1-\underline{\gamma_b})+(1-\overline{\gamma_b})n_j^\phi)(\underline{\gamma_b} + \overline{\gamma_b}n_j^\phi)}}.$$

or

$$1 > \frac{\left((1 - \underline{\gamma_b}) + (1 - \overline{\gamma_b})n_j^\phi \right) \left(\underline{\gamma_b} + \overline{\gamma_b}n_j^\phi \right)}{\left((1 - \underline{\gamma_a}) + (1 - \overline{\gamma_a})n_j^\phi \right) \left(\underline{\gamma_a} + \overline{\gamma_a}n_j^\phi \right)}.$$

Because we have assumed that n_0 is sufficiently large, $(\underline{\gamma_j} + \overline{\gamma_j}n_j^\phi) \approx \overline{\gamma_j}n_j^\phi$ and $\left((1 - \underline{\gamma_j}) + (1 - \overline{\gamma_j})n_j^\phi \right) \approx (1 - \overline{\gamma_j})n_j^\phi$. Thus it is enough to show that

$$1 > \frac{(1 - \overline{\gamma_b})n_j^\phi \overline{\gamma_b}n_j^\phi}{(1 - \overline{\gamma_a})n_j^\phi \overline{\gamma_a}n_j^\phi} = \frac{(1 - \overline{\gamma_b})\overline{\gamma_b}}{(1 - \overline{\gamma_a})\overline{\gamma_a}}.$$

But this follows because $.5 > \overline{\gamma_a} > \overline{\gamma_b}$.

B Appendix B: Data Appendix

Labor data from 1968-1991 comes from Current Population Surveys: March Individual Level Extracts, 1968-1992, Second ICPSR Version²¹. The remaining data is taken from the March Supplement of the Current Population Survey as made available through Data Ferret (dataferrett.census.gov).

Classification of industries is not consistent across time within or across these data sets. Table A1 below shows how we have reconciled the different classifications. For the 1968-1991 period, the variable used for industry categorization is v57 labeled “Industry.” For most years, this variable takes a value from 1 to 51. However, the industries associated with particular values varies through the years. For example, from 1968-1971, a value of 33 or 34 for v57 indicates an individual in the finance, insurance and real estate industry whereas from 1972-1982, this industry is indicated by a value of 35 or 36.

For 1991-2002 the variable indicating industry is A_DTIND. For 1992-2002 this variable is labeled “Current Status-Industry Detailed Recode” and for 2003-2004 it is labeled “Industry and Occupation-Main Job Detailed Industry.” Again the variable generally takes a value from 1 to 51.

In each year, data was adjusted using the appropriate population weights provide by the Current Population survey. Since data is provided sporadically, military is excluded.

Output data comes from the Bureau of Economic Analysis at the Department of Commerce and is available at http://www.bea.gov/beat/dn2/gdpbyind_data.htm.

²¹Chief Investigator: Robert Moffit, University of Michigan. Published by the Inter-university Consortium of Political and Social Research in 1999.

Table A.1 - Disaggregated Industry mapping

	1968- 1971	1972- 1982	1983- 1988	1989- 1991	1992- 2002	2003- 2004
Agriculture	1	1-2	1	1-2	1-2	1
Mining	2	3	2	3	3	3
Construction	3	4	3	4	4	4
Manufacturing-durable goods	4-14	6-17 ²²	4-17	4-18	5-18	5-13
Manufacturing-nondurable goods	5-24	18-27	18-27	19-28	19-28	14-20
Transportation	25-26	28-29	28	29	29	23
Communications	27	30	29	30	30	25-31
Utilities and sanitary services	28	31	30	31	31	24,39
Wholesale trade	29	32	31	32	32	21
Retail trade	30-31	33-34	32	33**	33	22,45-46
Finance, insurance and real estate	33-34	35-36	33-34	34-35	34-35	32-35
Private household	32	37	35	36	36	50
Business services	35	38	36	37	37	37-38
Personal services	37	40	38	39	39	48
Entertainment & rec. services	38	41	39	40	40	44
Hospitals	40	43	40	41	41	41
Medical services	39	42	41	42	42	42
Education	42	45	42	43	43	40
Social services	41	44	43	44	44	43
Other professional	43	46	44	45	45	36,49
Forestry and fisheries	44	47	45	46	46	2
Public administration	45-48	48-51	46	47-51	47-50	51
Auto and repair services	36	39	37	38	38	47

For economy of presentation, we aggregate the 23 industries of Table A.1 into the 13 industries of Table 1. This mapping is provided in Table A.2 below.

Table A.2 - Aggregated industry mapping

Aggregated industries in Table 1	Components from Table A.1
Educational and health services	Hospitals, medical services, education
Professional & business services	Business services, other professional , social services
Financial services	Finance, insurance and real estate
Public administration	Public administration
Leisure and hospitality services	Entertainment and recreational services
Information	Communications
Manufacturing	Manufacturing-durable goods, manufacturing-nondurable goods
Mining	Mining
Other services	Personal Services, utilities and sanitary services
Wholesale & retail trade	Retail trade, wholesale trade
Transportation	Transportation
Construction	Construction
Agriculture & forestry	Agriculture , forestry and fisheries

C Appendix C: Alternative cutoff for Table 1. Not intended for publication

Table 1 - Skilled to unskilled labor ratios across industries

	Skill to unskill ratio			Normalized skill to unskill ratio		
	1968 Value	2004 Value	Change	1968 Value	2004 Value	Change
Initially skilled						
Educational & health services	.562	.874	.312	1.000	1.556	.556
Professional & business services	.334	.682	.348	1.000	2.042	1.042
Financial services	.180	.582	.403	1.000	3.242	2.242
Public administration	.177	.622	.445	1.000	3.523	2.523
Leisure & hospitality services	.105	.487	.382	1.000	4.632	3.632
Information	.101	.636	.535	1.000	6.278	5.278
Initially unskilled						
Manufacturing	.083	.279	.197	1.000	3.384	2.384
Mining	.068	.159	.092	1.000	2.355	1.355
Other services	.061	.216	.155	1.000	3.546	2.546
Wholesale & retail trade	.056	.177	.122	1.000	3.186	2.186
Transportation	.046	.167	.121	1.000	3.607	2.607
Construction	.045	.111	.066	1.000	2.473	1.473
Agriculture & forestry	.026	.159	.134	1.000	6.251	5.251
Skilled aggregate	.334	.714	.380	1.000	2.137	1.137
Unskilled aggregate	.062	.190	.128	1.000	3.051	2.051

D Appendix D: Additional proofs not intended for publication.

D.1 This proves Proposition 1.

This a generalization of the proof in Blankenau and Cassou (2006) and closely follows the structure of that proof. Write the Hamiltonian for the consumer's problem as

$$\begin{aligned}
H = & e^{-\rho t} \ln \left[\frac{1}{n_a + n_b} \left[\sum_{j=a,b} \int_0^{n_j} x_{j,\omega}^{1-\psi} d\omega \right] \right]^{\frac{1}{1-\psi}} \\
& + \mu_{1t} (r_t k_{\iota,t} + w_t^s s_{\iota,t} + w_t^u u_{\iota,t} \\
& \quad + \sum_{j=a,b} \int_0^{n_j} [r_t k_{j,\omega} + w_t^s s_{j,\omega} + w_t^u u_{j,\omega}] d\omega \\
& \quad - \sum_{j=a,b} \int_0^{n_j} p_{j,\omega} x_{j,\omega} d\omega - (\delta + g_A + g_L) k_t) \\
& + \mu_{2t} \left(k_t - k_{\iota,t} - \sum_{j=a,b} \int_0^{n_j} k_{t,\omega} d\omega \right) \\
& + \mu_{3t} \left(1 - u_{\iota,t} - \sum_{j=a,b} \int_0^{n_j} u_{j,\omega} d\omega - s_{\iota,t} - \sum_{j=a,b} \int_0^{n_j} s_{j,\omega} d\omega - e_t \right) \\
& + \mu_{4t} \left(\theta e_t - s_{\iota,t} - \sum_{j=a,b} \int_0^{n_j} s_{j,\omega} d\omega \right),
\end{aligned}$$

and the transversality condition requires

$$\lim_{t \rightarrow \infty} \{\mu_{1t} k_t\} = 0.$$

Combining the first order conditions with (3), (4), (5) and the goods market clearance conditions, $y_{\iota,t} = i_t$, $y_{j,\omega} = x_{j,\omega}$ for $0 \leq j(\omega) \leq n_j$ and $j = a, b$, gives the following system of equations for the

competitive equilibrium.²³

$$H_{x_{j,\omega}} : \frac{e^{-\rho t}}{x_{j,\omega}^\psi \left[\sum_{j=a,b} \int_0^{n_j} x_{j,\omega}^{1-\psi} d\omega \right]} = \mu_{1t} p_{j,\omega} \quad (\text{D.1})$$

$$H_{k_t} : \mu_{2t} - (\delta + g_A + g_L) \mu_{1t} = -\mu_{1t} \quad (\text{D.2})$$

$$H_{u_{j,\omega}} : \mu_{1t} \frac{p_{j,\omega}(1-\alpha)(1-\gamma_{j,\omega})x_{j,\omega}u_{j,\omega}^{\sigma-1}}{\gamma_{j,\omega}s_{j,\omega}^\sigma + (1-\gamma_{j,\omega})u_{j,\omega}^\sigma} = \mu_{3t} \quad (\text{D.3})$$

$$H_{s_{j,\omega}} : \mu_{1t} \frac{p_{j,\omega}(1-\alpha)\gamma_{j,\omega}x_{j,\omega}s_{j,\omega}^{\sigma-1}}{\gamma_{j,\omega}s_{j,\omega}^\sigma + (1-\gamma_{j,\omega})u_{j,\omega}^\sigma} = (\mu_{3t} + \mu_{4t}) \quad (\text{D.4})$$

$$H_{k_{j,\omega}} : \mu_{1t} p_{j,\omega} \alpha \frac{x_{j,\omega}}{k_{j,\omega}} = \mu_{2t} \quad (\text{D.5})$$

$$H_{e_t} : \mu_{3t} = \theta \mu_{4t} \quad (\text{D.6})$$

$$H_{u_{\iota,t}} : \mu_{1t} \frac{i_t(1-\alpha)(1-\gamma_\iota)u_{\iota,t}^{\sigma-1}}{\gamma_\iota s_{\iota,t}^\sigma + (1-\gamma_\iota)u_{\iota,t}^\sigma} = \mu_{3t} \quad (\text{D.7})$$

$$H_{s_{\iota,t}} : \mu_{1t} \frac{i_t(1-\alpha)\gamma_\iota s_{\iota,t}^{\sigma-1}}{\gamma_\iota s_{\iota,t}^\sigma + (1-\gamma_\iota)u_{\iota,t}^\sigma} = (\mu_{3t} + \mu_{4t}) \quad (\text{D.8})$$

$$H_{k_{\iota,t}} : \mu_{1t} \alpha \frac{i_t}{k_{\iota,t}} = \mu_{2t} \quad (\text{D.9})$$

along with the constraints (1), (2), (8), (10), (11), and (12).

It is useful to organize the proof from here into three steps. In the first, we derive the time allocation expressions. In the second, we use these results to find dynamic expressions for capital and capital allocations. In the third we show convergence.

Step 1: We first derive expressions for other labor inputs as functions of $u_{j,\omega}$ and technology parameters. Using the notation j, ω' to indicate any $j, \omega' \in [0, n_j]$ for any $j = a, b$, equation (D.3) can be rearranged for goods $j(\omega)$ and $j(\omega)'$ to get

$$\frac{u_{j,\omega'}^{1-\sigma}}{u_{j,\omega}^{1-\sigma}} = \frac{p_{j,\omega'} x_{j,\omega'}}{p_{j,\omega} x_{j,\omega}} \frac{1-\gamma_{j,\omega'}}{1-\gamma_{j,\omega}} \frac{\gamma_{j,\omega} s_{j,\omega}^\sigma + (1-\gamma_{j,\omega}) u_{j,\omega}^\sigma}{\gamma_{j,\omega'} s_{j,\omega'}^\sigma + (1-\gamma_{j,\omega'}) u_{j,\omega'}^\sigma}. \quad (\text{D.10})$$

Similarly, (D.5) gives

$$\frac{p_{j,\omega'} x_{j,\omega'}}{p_{j,\omega} x_{j,\omega}} = \frac{k_{j,\omega'}}{k_{j,\omega}}. \quad (\text{D.11})$$

Put (D.11) into (D.10) and simplify to get

$$\frac{u_{j,\omega'}}{u_{j,\omega}} = \frac{k_{j,\omega'} (1-\gamma_{j,\omega'})}{k_{j,\omega} (1-\gamma_{j,\omega})} \frac{\gamma_{j,\omega} \left(\frac{s_{j,\omega}}{u_{j,\omega}} \right)^\sigma + (1-\gamma_{j,\omega})}{\gamma_{j,\omega'} \left(\frac{s_{j,\omega'}}{u_{j,\omega'}} \right)^\sigma + (1-\gamma_{j,\omega'})}. \quad (\text{D.12})$$

Use (D.3), (D.4) and (D.6) to show that

$$\frac{s_{j,\omega}}{u_{j,\omega}} = \left[\frac{\gamma_{j,\omega} \theta}{(1-\gamma_{j,\omega}) (1+\theta)} \right]^{\frac{1}{1-\sigma}}. \quad (\text{D.13})$$

²³We have added the notation on the left side of the equations to show which consumer first order conditions were used to arrive at each particular equation.

Put this into (D.12) and rearrange to derive

$$\frac{u_{j,\omega'}}{u_{j,\omega}} = \frac{k_{j,\omega'}}{k_{j,\omega}} \left[\frac{1 - \gamma_{j,\omega'}}{1 - \gamma_{j,\omega}} \right]^{\frac{1}{1-\sigma}} \frac{z_{j,\omega}^\sigma}{z_{j,\omega'}^\sigma} \quad (\text{D.14})$$

where $z_{j,\omega}$ is defined in the proposition. Combining (D.13) and (D.14) gives

$$\frac{u_{j,\omega'}}{s_{j,\omega}} = \frac{k_{j,\omega'}}{k_{j,\omega}} \left[\frac{1 - \gamma_{j,\omega'} \frac{1 + \theta}{\theta}}{\gamma_{j,\omega}} \right]^{\frac{1}{1-\sigma}} \frac{z_{j,\omega}^\sigma}{z_{j,\omega'}^\sigma}.$$

Next (D.3) and (D.7) give

$$\frac{i_t}{p_{j,\omega} x_{j,\omega}} = \frac{[\gamma_\iota s_{\iota,t}^\sigma + (1 - \gamma_\iota) u_{\iota,t}^\sigma]}{[\gamma_{j,\omega} s_{j,\omega}^\sigma + (1 - \gamma_{j,\omega}) u_{j,\omega}^\sigma]} \frac{(1 - \gamma_{j,\omega}) u_{\iota,t}^{1-\sigma}}{(1 - \gamma_\iota) u_{j,\omega}^{1-\sigma}}$$

while combining (D.5) and (D.9) gives

$$\frac{k_{\iota,t}}{k_{j,\omega}} = \frac{i_t}{p_{j,\omega} x_{j,\omega}}.$$

Combining the above two equations with (D.13) and its ι analogue gives

$$\frac{u_{\iota,t}}{u_{j,\omega}} = \frac{k_{\iota,t}}{k_{j,\omega}} \left[\frac{1 - \gamma_\iota}{1 - \gamma_{j,\omega}} \right]^{\frac{1}{1-\sigma}} \frac{z_{j,\omega}^\sigma}{z_\iota^\sigma} \quad (\text{D.15})$$

and similarly

$$\frac{s_{\iota,t}}{u_{j,\omega}} = \frac{k_{\iota,t}}{k_{j,\omega}} \left[\frac{\gamma_\iota \theta}{1 - \gamma_{j,\omega} \frac{1 + \theta}{\theta}} \right]^{\frac{1}{1-\sigma}} \frac{z_{j,\omega}^\sigma}{z_\iota^\sigma}. \quad (\text{D.16})$$

It is necessary to remove the capital expressions from (D.14)-(D.16). Toward this end, note that (D.1) yields

$$\frac{x_{j,\omega'}^\psi}{x_{j,\omega}^\psi} = \frac{p_{j,\omega}}{p_{j,\omega'}}. \quad (\text{D.17})$$

With (D.11) this is

$$\frac{k_{j,\omega'}}{k_{j,\omega}} = \frac{x_{j,\omega'}^{1-\psi}}{x_{j,\omega}^{1-\psi}}.$$

Using (2) and (D.13) and factoring out $\frac{u_{j,\omega'}}{u_{j,\omega}}$ this can be written as

$$\begin{aligned} \frac{k_{j,\omega'}}{k_{j,\omega}} &= \left[\frac{k_{j,\omega'}}{k_{j,\omega}} \right]^{\alpha(1-\psi)} \left(\frac{u_{j,\omega'}}{u_{j,\omega}} \right)^{(1-\alpha)(1-\psi)} \\ &\times \left[\frac{\gamma_{j,\omega'} \left[\frac{\gamma_{j,\omega'}}{1-\gamma_{j,\omega'}} \frac{\theta}{1+\theta} \right]^{\frac{\sigma}{1-\sigma}} + (1 - \gamma_{j,\omega'})}{\gamma_{j,\omega} \left[\frac{\gamma_{j,\omega}}{1-\gamma_{j,\omega}} \frac{\theta}{1+\theta} \right]^{\frac{\sigma}{1-\sigma}} + (1 - \gamma_{j,\omega})} \right]^{\frac{(1-\alpha)(1-\psi)}{\sigma}}. \end{aligned}$$

Using (D.14) this simplifies to

$$\frac{k_{j,\omega'}}{k_{j,\omega}} = \left[\frac{z_{j,\omega'}}{z_{j,\omega}} \right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)}{\psi}}. \quad (\text{D.18})$$

Given this expression for capital ratios, equations (D.14) and the subsequent equation simplify to

$$\frac{u_{j,\omega'}}{u_{j,\omega}} = \left[\frac{z_{j,\omega'}}{z_{j,\omega}} \right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} \left[\frac{1-\gamma_{j,\omega'}}{1-\gamma_{j,\omega}} \right]^{\frac{1}{1-\sigma}} \quad (\text{D.19})$$

and

$$\frac{u_{j,\omega'}}{s_{j,\omega}} = \left[\frac{1-\gamma_{j,\omega'}}{\gamma_{j,\omega}} \frac{1+\theta}{\theta} \right]^{\frac{1}{1-\sigma}} \left[\frac{z_{j,\omega'}}{z_{j,\omega}} \right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}}. \quad (\text{D.20})$$

We now need to remove the capital expression in (D.15) and (D.16). To do this, rearrange (D.15) and integrate each side to get

$$\sum_{j=a,b} \int_0^{n_j} \frac{k_{j,\omega}}{k_{l,t}} d\omega = \sum_{j=a,b} \int_0^{n_j} \frac{u_{j,\omega}}{u_{l,t}} \left[\frac{1-\gamma_l}{1-\gamma_{j,\omega}} \right]^{\frac{1}{1-\sigma}} \frac{z_{j,\omega}^\sigma}{z_l^\sigma} d\omega$$

or

$$\frac{1-\nu_t}{\nu_t} \frac{u_{l,t}}{(1-\gamma_l)^{\frac{1}{1-\sigma}}} z_l^\sigma = \sum_{j=a,b} \int_0^{n_j} u_{j,\omega} \left[\frac{1}{1-\gamma_{j,\omega}} \right]^{\frac{1}{1-\sigma}} z_{j,\omega}^\sigma d\omega.$$

Substitute in for $u_{j,\omega}$ using (D.14) and simplify to get

$$\frac{1-\nu_t}{\nu_t} \frac{u_{l,t} z_l^\sigma}{(1-\gamma_l)^{\frac{1}{1-\sigma}}} = \frac{u_{j,\omega'} z_{j,\omega'}^\sigma}{[1-\gamma_{j,\omega'}]^{\frac{1}{1-\sigma}}} \sum_{j=a,b} \int_0^{n_j} \frac{k_{j,\omega}}{k_{j,\omega'}} d\omega$$

which with (D.18) can be used to derive

$$u_{l,t} = \frac{\nu_t}{1-\nu_t} \frac{(1-\gamma_l)^{\frac{1}{1-\sigma}}}{(1-\gamma_{j,\omega})^{\frac{1}{1-\sigma}}} \frac{(Z_a + Z_b) u_{j,\omega}}{z_l^\sigma z_{j,\omega}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}}} \quad (\text{D.21})$$

where Z_j is defined in the proposition. Analogous algebra yields

$$s_{l,t} = \frac{\nu_t}{1-\nu_t} \left[\frac{\gamma_l}{1-\gamma_{j,\omega}} \frac{\theta}{1+\theta} \right]^{\frac{1}{1-\sigma}} \frac{(Z_a + Z_b) u_{j,\omega}}{z_l^\sigma z_{j,\omega}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}}}. \quad (\text{D.22})$$

We now use the time constraint to solve for $u_{j,\omega}$. Since $S_t = \theta e_t$, (12) can be written as

$$1 = \sum_{j=a,b} \int_0^{n_j} u_{j,\omega} d\omega + \left(\frac{1+\theta}{\theta} \right) \sum_{j=a,b} \int_0^{n_j} s_{j,\omega} d\omega + u_{l,t} + \left(\frac{1+\theta}{\theta} \right) s_l.$$

Using (D.19)-(D.22), this constraint gives

$$\begin{aligned} 1 &= \sum_{j=a,b} \int_0^{n_j} \left[\frac{z_{j,\omega}}{z_{j,\omega'}} \right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} \left[\frac{1-\gamma_{j,\omega}}{1-\gamma_{j,\omega'}} \right]^{\frac{1}{1-\sigma}} u_{j,\omega'} d\omega \\ &+ \left(\frac{1+\theta}{\theta} \right) \sum_{j=a,b} \int_0^{n_j} \left[\frac{\gamma_{j,\omega}}{1-\gamma_{j,\omega'}} \frac{\theta}{1+\theta} \right]^{\frac{1}{1-\sigma}} \left[\frac{z_{j,\omega}}{z_{j,\omega'}} \right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} u_{j,\omega} d\omega \\ &+ \frac{\nu_t}{1-\nu_t} \frac{(1-\gamma_l)^{\frac{1}{1-\sigma}}}{(1-\gamma_{j,\omega'})^{\frac{1}{1-\sigma}}} \frac{(Z_a + Z_b)}{z_{j,\omega'}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}}} u_{j,\omega'} \\ &+ \left(\frac{1+\theta}{\theta} \right) \frac{\nu_t}{1-\nu_t} \frac{1}{z_{j,\omega'}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}}} \left[\frac{\gamma_l}{1-\gamma_{j,\omega'}} \frac{\theta}{1+\theta} \right]^{\frac{1}{1-\sigma}} (Z_a + Z_b) u_{j,\omega'} \end{aligned}$$

or

$$\begin{aligned}
1 &= \sum_{j=a,b} \int_0^{n_j} \left[\frac{z_{j,\omega}}{z_{j,\omega'}} \right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} u_{j,\omega'} \\
&\quad \left[\left[\frac{1-\gamma_{j,\omega}}{1-\gamma_{j,\omega'}} \right]^{\frac{1}{1-\sigma}} + \left(\frac{1+\theta}{\theta} \right) \left[\frac{\gamma_{j,\omega}}{1-\gamma_{j,\omega'}} \frac{\theta}{1+\theta} \right]^{\frac{1}{1-\sigma}} \right] d\omega \\
&\quad + \frac{\nu_t}{1-\nu_t} \frac{1}{z_{j,\omega'}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} z_l^\sigma} (Z_a + Z_b) u_{j,\omega'} \frac{1}{[1-\gamma_{j,\omega'}]^{\frac{1}{1-\sigma}}} \\
&\quad \left[(1-\gamma_l)^{\frac{1}{1-\sigma}} + \left(\frac{1+\theta}{\theta} \right) \left[\gamma_l \frac{\theta}{1+\theta} \right]^{\frac{1}{1-\sigma}} \right]
\end{aligned}$$

which can be written as

$$\begin{aligned}
1 &= \frac{u_{j,\omega'}}{[1-\gamma_{j,\omega'}]^{\frac{1}{1-\sigma}}} \sum_{j=a,b} \int_0^{n_j} \left[\frac{z_{j,\omega}}{z_{j,\omega'}} \right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\psi\sigma}{\psi}} z_{j',t}(j(\omega))^\sigma d\omega \\
&\quad + \frac{\nu_t}{1-\nu_t} \frac{1}{z_{j,\omega'}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\psi\sigma}{\psi}} z_l^\sigma} (Z_a + Z_b) u_{j,\omega'} \frac{1}{[1-\gamma_{j,\omega'}]^{\frac{1}{1-\sigma}}} z_l^\sigma
\end{aligned}$$

and finally simplified to

$$1 = u_{j,\omega'} \frac{Z_a + Z_b}{z_{j,\omega'}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\psi\sigma}{\psi}} [1-\gamma_{j,\omega'}]^{\frac{1}{1-\sigma}}} \left[1 + \frac{\nu_t}{1-\nu_t} \right].$$

As this relationship holds for any good, we can drop the prime and solve for $u_{j,\omega}$ as given in (18). Then substituting this expression into (D.20)-(D.22) gives (19)-(17).

Step 2. We now derive the goods market results. (D.1) will be used to analyze the dynamics. Dynamics are expressed in terms of ν_t , k_t and the underlying parameters. Thus the first task is to solve for $x_{j,\omega}$ and $p_{j,\omega}$ in terms of these items so that they can be eliminated in (D.1). Using (18) and (19) in (2) and simplifying gives

$$y_{j,\omega} = k_{j,\omega}^\alpha \left(\frac{(1-\nu_t) z_{j,\omega}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)+(1-\sigma)\psi}{\psi}}}{Z_a + Z_b} \right)^{1-\alpha}. \quad (\text{D.23})$$

To eliminate $k_{j,\omega}$ from this expression, use (D.18) and the definition of ν_t to get

$$(1-\nu_t) k_t = \sum_{j=a,b} \int_0^{n_j} k_{j,\omega} d\omega = \sum_{j=a,b} \int_0^{n_j} \left[\frac{z_{j,\omega}}{z_{j,\omega'}} \right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)}{\psi}} k_{j,\omega'} d\omega$$

which can be solved to get

$$k_{j,\omega'} = \frac{(1-\nu_t) z_{j,\omega'}^{\frac{(1-\alpha)(1-\psi)(1-\sigma)}{\psi}} k_t}{Z_a + Z_b}.$$

As this holds for all products, the prime can be dropped. Furthermore, since market clearing requires $y_{j,\omega} = x_{j,\omega}$, putting the above expression into (D.23) and rearranging gives

$$x_{j,\omega} = \frac{z_{j,\omega}^{\frac{(1-\alpha)(1-\sigma)}{\psi}} (1-\nu_t) k_t^\alpha}{Z_a + Z_b}. \quad (\text{D.24})$$

To solve for prices, note (D.5) and (D.9) give

$$\frac{k_t}{k_{j,\omega}} = \frac{i_t}{p_{j,\omega} x_{j,\omega}}.$$

Using equations (2) and (1), this is

$$\frac{k_{t,t}}{k_{j,\omega}} = \frac{k_{t,t}^\alpha [\gamma_t s_{t,t}^\sigma + (1 - \gamma_t) u_{t,t}^\sigma]^{\frac{1-\alpha}{\sigma}}}{p_{j,\omega} k_{j,\omega}^\alpha [\gamma_{j,\omega} s_{j,\omega}^\sigma + (1 - \gamma_{j,\omega}) u_{j,\omega}^\sigma]^{\frac{1-\alpha}{\sigma}}}.$$

Then using equations (14), (15) and (18)-(17) in the above expression, simplifying and solving for $p_{j,\omega}$ it can be shown that

$$p_{j,\omega} = \frac{z_t^{(1-\sigma)(1-\alpha)}}{z_{j,\omega}^{(1-\sigma)(1-\alpha)}}. \quad (\text{D.25})$$

We now consider the dynamics. Putting equations (D.24) and (D.25) into (D.1) and simplifying gives

$$\frac{1}{\mu_{1t}} = e^{\rho t} \left[z_t^{(1-\sigma)(1-\alpha)} \right] \left[\frac{(1 - \nu_t) k_t^\alpha}{Z_a + Z_b} \right] \sum_{j=a,b} \int_0^{n_j} z_{j,\omega'}^{\frac{(1-\alpha)(1-\sigma)(1-\psi)}{\psi}} d\omega'$$

Using the definition of $Z_a + Z_b$ in the proposition this is

$$\frac{1}{\mu_{1t}} = e^{\rho t} z_t^{(1-\sigma)(1-\alpha)} (1 - \nu_t) k_t^\alpha.$$

Taking the natural log and time differential of each side gives (since $1 - \nu_t = -\dot{\nu}_t$)

$$-\frac{\dot{\mu}_{1t}}{\mu_{1t}} = \rho + \frac{-\dot{\nu}_t}{1 - \nu_t} + \alpha \frac{\dot{k}_t}{k_t}. \quad (\text{D.26})$$

Using (8) with the goods market clearance conditions, $y_{t,t} = i_t$, $y_{j,\omega} = x_{j,\omega}$ for $0 \leq j(\omega) \leq n_j$ and $j = a, b$ as well as the input market prices, (3), (4), (5), gives

$$i_t = \dot{k}_t + (\delta + g_A + g_L) k_t. \quad (\text{D.27})$$

Using (D.2), (D.9) and (D.27), (D.26) becomes

$$\alpha \frac{\dot{i}_t}{k_{t,t}} - (\delta + g_A + g_L) = \rho + \frac{-\dot{\nu}_t}{1 - \nu_t} + \alpha \left[\frac{\dot{i}_t}{k_t} - (\delta + g_A + g_L) \right]. \quad (\text{D.28})$$

Use (1), (14), (16) and (17) to get

$$i_t = \nu_t k_t^\alpha z_t^{(1-\sigma)(1-\alpha)}$$

which implies

$$\frac{\dot{i}_t}{k_t} = \nu_t k_t^{\alpha-1} z_t^{(1-\sigma)(1-\alpha)} \quad \text{or} \quad \frac{\dot{i}_t}{k_{t,t}} = k_t^{\alpha-1} z_t^{(1-\sigma)(1-\alpha)}. \quad (\text{D.29})$$

This can be substituted into (D.28) and rearranged to get (20). To derive (21), use (D.29) to rewrite (D.27).

Step 3. The proof of global stability is precisely as in Blankenau and Cassou (2006).