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"A Monetary Approach to Asset Liquidity"

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A Monetary Approach to Asset Liquidity*

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Abstract

This paper offers a monetary theory of asset liquidity—one that emphasizes the role of assets in payment arrangements—and it explores the implications of the theory for the relationship between assets' intrinsic characteristics and liquidity, and the effects of policy on asset prices and welfare. The environment is a random-matching economy in which risk-free bonds coexist with a risky asset, equity, and no restrictions are imposed on payment arrangements. The liquidity differential between bonds and equity results from an informational asymmetry regarding the fundamental value of equity. The model predicts that the risk-free asset is a strictly preferred means of payment, while equity is partially illiquid. As a consequence, the risk-free rate is below its fundamental value, and the equity premium is positive, provided that the supply of bonds is not too large. This result holds irrespective of the supply of equity and despite agents being risk-neutral. Moreover, the equity premium tends to increase as equity becomes riskier. Finally, an increase in the supply of bonds has a permanent liquidity effect. It raises the risk-free rate, output, and welfare, and it reduces the equity premium.

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J.E.L. Classification: D82, D83, E40, E50

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1 Introduction

Liquidity considerations matter for macroeconomics. They help explain asset pricing anomalies, the codetermination of asset prices and macroeconomic conditions, and the transmission mechanism of monetary policy.[1] Since liquidity can have different meanings, I will define an asset as illiquid if it can be sold at short notice only for a discounted price, or not at all.[2] Liquid assets have an essential role in economies where credit arrangements that would allow households and firms to finance spending shocks (e.g., consumption or investment opportunities) are not always feasible. A critical observation from Kiyotaki and Moore (2005) and Lagos (2006) is that not all assets are equally suitable for helping agents face these shocks: some assets are more liquid than others. In Kiyotaki and Moore, agents who hold land and capital can use only a fraction of their capital stock to finance investment opportunities. In Lagos, agents hold risk-free bonds and equity, but equity shares can only be used to finance a fraction of their consumption opportunities. While these liquidity differences among assets help explain several macroeconomic phenomena, the differences in liquidity themselves are left unexplained by the proposed theories. In particular, no link is made between the characteristics of an asset, such as its degree of risk, the supply of the asset, to the ease with which it is traded.

The aim of this paper is to provide a monetary theory of asset liquidity—one that emphasizes the role of assets in payment arrangements—and to explore the implications of the theory for the relationship between assets' intrinsic characteristics and liquidity, and the effects of policy on asset prices and welfare. Following the literature pioneered by Kiyotaki and Wright (1989), this paper considers economies in which trading opportunities between randomly-matched, anonymous agents makes the use of some assets essential as means of payment. I consider an economy where two assets can serve this role, risk-free bonds and risky equity. (I also consider fiat money in a later part of the paper.) Without additional frictions beyond the ones that rule out credit arrangements, all forms of wealth are equally good as means of payment, and agents are indifferent between which asset they spend or which they accept. In order to overcome this indeterminacy, I assume that the liquidity differential between bonds and equity stems from an informational asymmetry regarding the fundamental value of the equity. Specifically, agents paying with an asset are better informed about its future performance than agents who receive it, which makes it costly to trade.

[1] See, e.g., Marshall (1992), Bansal and Coleman (1996), Kiyotaki and Moore (2005), and Lagos (2006).
[2] I thank Steve LeRoy for suggesting this specific definition to me.

A key insight of the theory is that the risk-free asset is a strictly preferred means of payment, and equity is partially illiquid. In order to finance their consumption opportunities, individuals deplete their bond holdings first, and they use equity as a last resort. Moreover, individuals retain a fraction of their equity holdings even when their consumption is inefficiently low. A major insight of Kiyotaki and Wright (1989) was to show that the acceptability of a good depends on its storage cost as well as other fundamentals (e.g., the pattern of specialization) and beliefs. In the same vein, I find that the liquidity of the equity, as captured by the quantity of equity agents can sell before experiencing a deterioration in the terms of trade, depends on its dividend process. Equity becomes more illiquid as the dispersion of the dividends across states increases. In the limiting case where the equity has no value in some states, it becomes fully illiquid and, in the absence of risk-free assets, trades shut down.

Bansal and Coleman (1996) and Lagos (2006) showed that liquidity differences among assets can explain seemingly anomalistic asset prices. Similarly, my model predicts that the risk-free rate is below agents' discount rate provided that the supply of bonds is not too large, even if equity is abundant. Moreover, the liquidity differential between bonds and equity generates an equity premium irrespective of the economy-wide stock of equity, and despite agents being risk-neutral with respect to the consumption of the dividend good. The size of the equity premium depends on the riskiness of equity: it tends to increase as the equity becomes riskier. I also show that the private-information friction raises the price of risk-free bonds and the price difference between bonds and equity relative to their complete-information counterparts.

By taking into account the role of assets' in payments, the model provides a channel through which policy–described as a change in the supply of the risk-free bonds–affects asset prices, the structure of asset returns, and output. If the quantity of bonds is below a threshold, an increase in the supply of bonds can accommodate a larger demand for liquid assets, which raises the risk-free rate, output, and welfare. Moreover, if there is a shortage of wealth in low-dividend states, then an increase in the supply of bonds increases the rate of return of equity, and decreases the equity premium. This liquidity effect of policy is permanent, and it can account for long-lasting effects of a change in the level of public debt. The optimal policy is such that the demand for risk-free assets is satiated. In that case, asset prices are driven down to their fundamental values, and equity is illiquid, i.e., its transaction velocity (in some states) is zero.

Finally, I introduce fiat money along with bonds and equity. Following Lester, Postlewaite, and Wright (2007), I assume that equity and bonds can be counterfeited at no cost, and can only authenticated by a fraction of agents. The model predicts a negative relationship between inflation and equity's expected return. If the supply of risk-free bonds is below a threshold, then an open-market operation has a permanent liquidity effect on the risk-free rate, output, and welfare. Finally, the model can generate both a rate-of-return differential between risk-free bonds and fiat money, and an equity premium in an economy with risk-neutral agents.

The paper is organized as follows. Section 1.1 provides a review of the relevant literature. The environment is described in Section 2, and the social optimum is characterized in Section 3. Section 4 analyzes the bargaining game under incomplete information. Section 5 embeds the bargaining game into a general equilibrium structure and studies the effects of policy and fundamentals on asset liquidity. Finally, Section 6 introduces fiat money.

1.1 Related literature

A distinctive feature of my environment is the presence of multiple assets traded in bilateral meetings under private information. Similarly, Hopenhayn and Werner (1996) study a three-period nonmonetary game with indivisible assets. Velde, Weber, and Wright (1999) consider a model with two indivisible commodity monies to account for Gresham's law. Golosov, Lorenzoni, and Tsyvinski (2008) focus on the transmission of information through decentralized trading and its implications for long-run efficiency and the value of information.

Private information frictions are omnipresent in both the finance and the monetary literature. Asymmetries of information are used to endogenize transaction costs in financial markets (e.g., Kyle, 1985; Glosten and Milgrom, 1985), security design (e.g., DeMarzo and Duffie, 1999), and capital structure choices (e.g., Myers and Majluf, 1984). The monetary literature has resorted to private-information problems to explain the role of money when goods are of unknown quality (e.g., Williamson and Wright, 1994; Banarjee and Maskin, 1996) and when individuals have some private information about their ability to repay their debt (e.g., Jafarey and Rupert, 2001).[3]

The private-information problem in my model provides a foundation for the trading restrictions that have been imposed in some recent models that have fiat money coexisting with other assets, e.g., Aruoba and

[3] Berentsen and Rocheteau (2004) introduce a moral hazard similar to Williamson and Wright (1994) into a model with divisible money. The "counterfeit" consumption good is perishable, it has no value, and only a pooling mechanism is considered. Li (1995) constructs a search model, in which there is uncertainty about the quality of commodity monies.

Wright (2003), Aruoba, Waller, and Wright (2007), Berentsen, Menzio, and Wright (2007), Lagos (2006), and Telyukova and Wright (2008).[4] Aiyagari, Wallace, and Wright (1996), Wallace (1996, 2000), and Cone (2005) emphasize asset divisibility, or lack of divisibility, to explain the coexistence of money and interest-bearing assets, and the liquidity structure of asset yields. Following Freeman (1985), Lester, Postlewaite, and Wright (2007) and Kim and Lee (2008) explain the illiquidity of capital goods by the assumption that claims on capital can be costlessly counterfeited and can only be authenticated in a fraction of meetings. In Lester *et al.*, this fraction of meetings is endogenized by assuming that agents can invest in a technology to recognize claims on capital. Rocheteau (2009b) investigates the case in which counterfeits are produced at a positive cost and shows that the lack of recognizability manifests itself by an endogenous upper bound on the transfer of assets in uninformed matches. Kiyotaki and Moore (2005) assume that the transfer of ownership of capital is not instantaneous so that an agent can steal a fraction of his capital before the transfer is effective.[5] Similarly, Holmstrom and Tirole (1998, 2001) develop a corporate finance approach to liquidity, where a moral hazard problem prevents claims on corporate assets from being written.

Shi (1999) and Aruoba and Wright (2003) were the first to introduce capital goods into search-theoretic models with unrestricted asset holdings. However, capital goods could not be used as means of payment in decentralized trades. Lagos and Rocheteau (2008) relax this assumption and let money and capital compete as media of exchange. Geromichalos, Licari, and Suarez-Lledo (2007) study a similar model with a fixed stock of capital. The environment in this paper is closer to the one in Lagos (2006) in which fiat money is replaced by risk-free bonds and capital is a risky asset. In contrast to my model, there is no informational asymmetries and, for calibration purposes, the use of capital goods as means of payment is restricted in a fraction of the meetings.[6] Finally, Nosal and Rocheteau (2009) show that a standard search-theoretic monetary model can generate rate-of-return differences among seemingly identical assets without imposing

[4] Aruoba and Wright (2003) and Aruoba, Waller, and Wright (2007) also refer to the lack of portability of capital goods to justify the assumption that capital cannot be used as means of payment in decentralized markets. Telyukova and Wright (2008, Section 4) lay down an extension of their model with "Lucas trees," in which agents pay a fixed cost if they use their real assets as means of payment.

[5] Kiyotaki and Moore (2005) provide an alternative explanation for why capital may not be perfectly liquid: "there may be different qualities of capital, and buyers may be less informed than sellers so that there is adverse selection in the second-hand market." This is the avenue I follow in this paper. Similarly, Zhu (2008, Section 4) discusses how one could introduce capital into his overlapping-generations model with search, and he argues that to maintain the transaction role of money, "one could assume some private information about the quality of capital, similar to the private information problem on the quality of goods in Williamson and Wright (1994)."

[6] Lagos (2006) studies first a version of the model where the use of Lucas' trees as means of payment is unrestricted. The trading restriction is introduced to allow the model to match the equity premium in the data. Shi (2004) adopts a similar assumption in a search model with fiat money and nominal bonds.

trading restrictions and without violating Pareto-efficiency in bilateral trades.

2 Environment

The environment is similar to the one in Lagos and Wright (2005) and Rocheteau and Wright (2005). Time is discrete, starts at $t = 0$, and continues forever.[7] Each period has two subperiods: a morning, where trades occur in a decentralized market (DM), followed by an afternoon, where trades take place in a competitive market (CM). There is a continuum of infinitely-lived agents divided evenly into two types, *buyers* and *sellers*, who differ in terms of when they produce and consume. The labels *buyers* and *sellers* indicate agents' roles in the DM. Let $\mathcal{B}$ denote the set of buyers, $\mathcal{S}$ the set of sellers, and $\mathcal{J} = \mathcal{B} \cup \mathcal{S}$. There are two perishable consumption goods, one produced in the DM and the other in the CM.

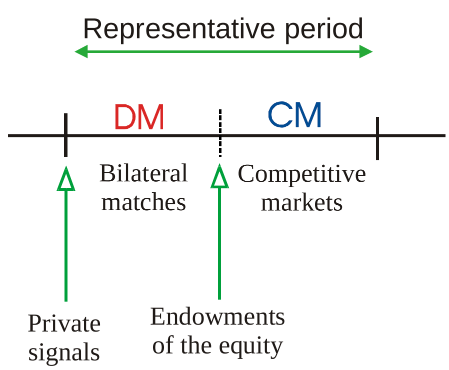

Figure 1: Timing

Buyers and sellers can produce and consume in the CM. In the DM, however, buyers receive an opportunity to consume, while sellers have an opportunity to produce. The lifetime expected utility of a buyer from date 0 onward is

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t \left[x_t - \ell_t + u(y_t) \right], \tag{1}$$

where x_t is the CM consumption of period t, ℓ_t is the CM disutility of work, y_t is the DM consumption, and $\beta \in (0, 1)$ is a discount factor. The utility function $u(y)$ is twice continuously differentiable, $u(0) = 0$, $u'(0) = \infty$, $u'(y) > 0$, and $u''(y) < 0$. The production technology in the CM, f, is linear with labor as the only input, $f(\ell_t) = \ell_t$.

[7] The assumption of an infinite time horizon is not necessary for most of the paper. It is needed, however, in Section 6 when I introduce fiat money.

The lifetime expected utility of a seller from date 0 onward is

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^{t}\left[x_{t}-\ell_{t}-c(y_{t})\right], \tag{2}$$

where y_t is the DM production. The cost function $c(y)$ is twice continuously differentiable, $c(0) = c'(0) = 0$, $c'(y) > 0$, $c''(y) \geq 0$, and $c(y) = u(y)$ for some $y > 0$. Let y^* denote the solution to $u'(y^*) = c'(y^*)$.

At the beginning of the CM, each buyer is endowed with $A > 0$ units of a one-period-lived real asset called equity. Because of the absence of wealth effects, it is irrelevant for the allocations who receives the endowment of equity. Equity is divisible and uncounterfeitable. Each unit of the period-t equity yields κ_{t+1} units of CM output, which is delivered in the CM of $t+1$, and it fully depreciates subsequently. The dividend shocks, κ_t, are independent across time, with $\pi_h = \Pr[\kappa_t = \kappa_h] \in (0,1)$, $\pi_\ell = \Pr[\kappa_t = \kappa_\ell] = 1 - \pi_h$, and $0 < \kappa_\ell < \kappa_h$. With no loss in generality, I normalize the expected dividend, $\pi_h\kappa_h + \pi_\ell\kappa_\ell$, to one.[8]

The government supplies a constant quantity, Z, of one-period-lived, risk-free bonds. The safety of bonds is backed by the unrestricted ability of the government to tax agents in the CM.[9] Bonds are perfectly divisible, and each unit pays one unit of output in the CM. In Section 6, I also consider the case of fiat currency, an infinitely-lived, intrinsically useless object supplied by the government. Policy is described as the buying and selling of risk-free bonds by the government. The interest payments are financed by lump-sum taxes to buyers in the CM.

In the CM, there is a competitive market in which agents can trade consumption goods, bonds, and equity shares. In the DM, each seller is matched bilaterally with a buyer drawn at random from $\mathcal{B}$. The buyer makes an offer that the seller accepts or rejects. If the offer is accepted, the trade is implemented.[10] All trades in the DM are *quid pro quo.* Matched agents can transfer any nonnegative quantity of DM output

[8] In Rocheteau (2008, Appendix D) I show that the model can be generalized to allow for more than two realizations for the dividend shock. Also, the case where assets are long-lived complicates significantly the proof for the uniqueness of the equilibrium and other analytical results, but it does not affect the main insights. It is nonetheless a worthwhile extension to investigate asset-price fluctuations in the business cycle. Finally, it would be equivalent to consider idiosyncratic dividend shocks as long as buyers cannot fully diversify the risk of their portfolios.

[9] The government can force buyers to pay taxes in the CM, but it has no enforcement power in the DM, and it does not observe agents' trading histories. In a related model, Andolfatto (2007) considers the case where the government has limited coercion power–it cannot confiscate output and cannot force agents to work–and the payment of lump-sum taxes is voluntary: agents can avoid paying taxes by not participating in the CM. He shows that if agents are sufficiently impatient, then the optimal monetary policy is not incentive-feasible.

[10] I chose a bargaining protocol where the buyer makes a take-it-or-leave-it offer because it has been extensively used in monetary theory, and it remains tractable under private information. Alternatively, I could have let sellers set the terms of trade. Because of the informational asymmetry, buyers in the low dividend state would be able to capture a rent proportional to the transfer of equity in the high dividend state. See Ennis (2008) for a study of this trading mechanism in a related environment. However, in my environment buyers would have no incentives to hold bonds or fiat money because it would reduce the size of their surplus in the low dividend state.

and any quantity of their asset holdings. Credit arrangements are not incentive-feasible since agents are anonymous and cannot commit.

An informational asymmetry about the value of equity is introduced as follows. Buyers who enter the DM in period t receive a perfectly informative signal about the dividend of the equity, κ_t. Sellers, in contrast, only learn the realization of the dividend in the CM of the same period.[11] An advantage in terms of formalization of having equity traded in bilateral meetings in the DM, besides being a realistic feature of many asset markets, is that it prevents the price from revealing the buyers' information at no cost.

3 Social optimum

Consider a social planner who chooses an allocation in order to maximize the sum of the lifetime expected utilities of all agents in the economy. The planner has full command over the economy's resources, but it has no private information about the future value of equity, i.e., it observes the realization of the dividend shock, κ_t, at the beginning of the CM in period t.

Let $\mathcal{M}_t$ denote the set of bilateral matches (j, j') composed of one buyer, $j \in \mathcal{B}$, and one seller, $j' \in \mathcal{S}$, in the DM of period t. The expression for social welfare is then

$$\mathcal{W} = \sum_{t \geq 0} \beta^t \int_{j \in \mathcal{J}} [x_t(j) - \ell_t(j)]\, dj + \sum_{t \geq 0} \beta^t \int_{(j,j') \in \mathcal{M}_t} \{u[y_t(j)] - c[y_t(j')]\}\, d(j, j'). \quad (3)$$

The first integral on the right side of (3) corresponds to the consumption net of the disutility of work for all agents from $t = 0$ onwards. The second term is buyers' consumption net of sellers' disutility of production in bilateral matches formed in the DM. The planner is subject to the following feasibility constraints:

$$\int_{j \in \mathcal{J}} x_t(j) dj \leq \int_{j \in \mathcal{J}} \ell_t(j) dj + A\kappa_t, \quad \forall t \geq 0 \quad (4)$$

$$y_t(j) \leq y_t(j'), \quad \forall (j, j') \in \mathcal{M}_t, \quad \forall t \geq 0. \quad (5)$$

Feasibility constraint (4) requires agents' CM consumption in period t to be at most equal to the aggregate production in that period, including the output generated by equity, $\kappa_t A$. Feasibility condition (5) indicates

[11] There are several ways one can interpret this informational asymmetry. One can think of a seller as consolidating the roles of a dealer of assets and a producer. In accordance with the market microstructure literature, the dealer is uninformed about the future value of the asset (e.g., Glosten and Milgrom, 1985). Alternatively, one could adopt the assumption of Plantin (2008) that agents acquire some private information about the value of an asset by holding it. This assumption is relevant for assets that are not traded publicly, such as securitized pools of loans. In my model, sellers have no strict incentives to hold the asset, even if they could learn its future dividend in the DM, while buyers have a liquidity motive to hold the asset. Alternative information structures could be considered, e.g., a fraction of buyers and a fraction of sellers are informed. Provided that buyers know whether sellers are informed or not, the model remains tractable.

that the buyer's consumption in a bilateral match is no greater than the seller's production in that match. The planner's problem can be rewritten as a sequence of static problems, i.e.,

$$\max_{x_t,\ell_t} \int_{j\in\mathcal{J}} [x_t(j) - \ell_t(j)]\, dj \quad \text{s.t. (4)} \tag{6}$$

$$\max_{y_t} \int_{(j,j')\in\mathcal{M}_t} \{u[y_t(j)] - c[y_t(j')]\}\, d(j,j') \quad \text{s.t. (5)} \tag{7}$$

The planner is indifferent about how the CM goods are allocated between agents. The optimal consumption and production in bilateral matches satisfy $y_t(j) = y_t(j') = y^*$ for all $(j,j') \in \mathcal{M}_t$.

4 Payments under private information

In this section, I consider the bargaining game between a buyer holding a portfolio composed of a^b units of equity and z^b bonds, and a seller with a portfolio of a^s units of equity and z^s bonds. The analysis of the bargaining game is simplified by assuming that the buyer's and seller's portfolios are common knowledge in the match.[12]

In order to define the payoffs in the bargaining game, it is useful to derive first some properties of the value functions in the CM. Let $W^b(z,a,\kappa)$ denote the value function of a buyer at the end of the DM (before the CM opens) holding z units of bonds and a units of equity, when the dividend state is $\kappa \in \{\kappa_\ell, \kappa_h\}$.

$$W^b(z,a,\kappa) = \max_{x,\ell,z',a'} \left\{x - \ell + \beta\mathbb{E}V^b\left(z',a',\kappa'\right)\right\} \tag{8}$$

$$\text{s.t.} \quad x + q_z z' + q_a a' + T = \ell + \kappa a + z + q_a A, \tag{9}$$

where $V^b(z,a,\kappa)$ is the value function of the buyer at the beginning of the DM, q_z is the price of bonds (expressed in CM output), q_a is the price of equity, and $T \equiv Z(1-q_z)$ is the lump-sum tax (or transfer if $T < 0$) by the government. The expectation is taken with respect to the future dividend state κ'. According to (8), each buyer chooses his net consumption, $x - \ell$, and his portfolio, z' and a', in order to maximize his expected lifetime utility subject to the budget constraint (9). According to (9), the value of the buyer's initial portfolio in terms of CM output is $\kappa a + z$. In order to hold a portfolio (z', a') in the next CM, the buyer must invest $q_z z'$ of current output in bonds and $q_a a'$ in equity shares. He must also pay some lump-sum

[12] This assumption is made in order to avoid having to specify the agents' beliefs regarding the portfolio held by their partners in the matches. It will be shown in the following that the surplus functions in the DM are weakly monotone increasing in the agents' asset holdings. Hence, if agents had the possibility of showing their portfolios in a pre-stage of the bargaining game, there would be an equilibrium in which they would do so truthfully.

taxes T, and he receives an endowment of equity worth $q_a A$. Substitute $x - \ell = \kappa a + z - q_z z' + q_a(A - a') - T$ from (9) into (8) to obtain

$$W^b(z,a,\kappa) = \kappa a + z + q_a A - T + \max_{z',a'} \left\{ -q_z z' - q_a a' + \beta \left[\pi_h V^b(z',a',\kappa_h) + \pi_\ell V^b(z',a',\kappa_\ell) \right] \right\}. \quad (10)$$

The buyer's value function in the CM is linear in his wealth. Moreover, a buyer's portfolio choice is independent of his initial portfolio when he entered the period. Both properties greatly simplify the model. By the same reasoning, the expected lifetime utility of a seller at the end of the DM is given by

$$W^s(z,a,\kappa) = \kappa a + z + \max_{z',a'} \left\{ -q_z z' - q_a a' + \beta V^s(z',a') \right\}, \quad (11)$$

where $V^s(z,a)$ is the value function of the seller upon entering the DM.

The bargaining game between the buyer and the seller has the structure of a signaling game.[13] A strategy for the buyer specifies an offer $(y,d,\tau) \in \mathcal{F} \equiv \mathbb{R}_+ \times [-a^s, a^b] \times [-z^s, z^b]$, where y is the output produced by the seller, d is the transfer of equity by the buyer, and τ is the transfer of bonds, as a function of the buyer's type (i.e., his private information about the future dividend of the equity). The transfers of assets are constrained by the buyer's and seller's portfolios. A strategy for the seller is an acceptance rule that specifies a set $\mathcal{A} \subseteq \mathcal{F}$ of acceptable offers.

The buyer's payoff in the dividend state κ is

$$\left[u(y) + W^b\left(z^b - \tau, a^b - d, \kappa\right)\right] \mathbb{I}_{\mathcal{A}}(y,d,\tau) + W^b\left(z^b, a^b, \kappa\right)\left[1 - \mathbb{I}_{\mathcal{A}}(y,d,\tau)\right],$$

where $\mathbb{I}_{\mathcal{A}}(y,d,\tau)$ is an indicator function that is equal to one if $(y,d,\tau) \in \mathcal{A}$. If an offer is accepted, then the buyer enjoys his utility of consumption in the DM, $u(y)$, but he forgoes d units of equity and τ units of bonds. Using the linearity of the buyer's value function, and omitting the constant terms, the buyer's payoff can be expressed as his surplus, $[u(y) - \kappa d - \tau]\, \mathbb{I}_{\mathcal{A}}(y,d,\tau)$.

Similarly, the seller's (Bernoulli) payoff function is

$$\left[-c(y) + W^s\left(z^s + \tau, a^s + d, \kappa\right)\right] \mathbb{I}_{\mathcal{A}}(y,d,\tau) + W^s\left(z^s, a^s, \kappa\right)\left[1 - \mathbb{I}_{\mathcal{A}}(y,d,\tau)\right],$$

and his surplus is $[-c(y) + \kappa d + \tau]\, \mathbb{I}_{\mathcal{A}}(y,d,\tau)$. In order to accept or reject an offer, the seller will have to form expectations about the dividend of the equity. Let $\lambda(y,d,\tau) \in [0,1]$ represent the updated belief of

[13] See Appendix B in Rocheteau (2009a) for a more detailed presentation of signaling games.

a seller that the buyer holds a high-dividend asset ($\kappa = \kappa_h$), conditional on the offer (y, d, τ) being made. Then, $\mathbb{E}_\lambda [\kappa] = \lambda(y, d, \tau)\kappa_h + [1 - \lambda(y, d, \tau)] \kappa_\ell$.

For a given belief system, the set of acceptable offers for a seller is

$$\mathcal{A}(\lambda) = \{(y, d, \tau) \in \mathcal{F} : -c(y) + \{\lambda(y, d, \tau)\kappa_h + [1 - \lambda(y, d, \tau)] \kappa_\ell\} d + \tau \geq 0\}. \tag{12}$$

For an offer to be acceptable, the seller's disutility of production in the DM, $-c(y)$, must be compensated for by his expected utility in the next CM, $\mathbb{E}_\lambda [\kappa] d + \tau$. I adopt a tie-breaking rule according to which a seller agrees to any offer that makes him indifferent between accepting or rejecting a trade.[14] The problem of a buyer holding an asset of quality κ is then

$$\max_{y,d,\tau} [u(y) - \kappa d - \tau] \mathbb{I}_{\mathcal{A}}(y, d, \tau) \quad \text{s.t.} \quad (y, d, \tau) \in \mathbb{R}_+ \times [-a^s, a^b] \times [-z^s, z^b]. \tag{13}$$

The equilibrium concept is perfect Bayesian equilibrium. An equilibrium of the bargaining game is a profile of strategies for the buyer and the seller, and a belief system λ. If (y, d, τ) is an offer made in equilibrium, then $\lambda(y, d, \tau)$ is derived from the seller's prior belief according to Bayes's rule. Since there is no discipline for out-of-equilibrium beliefs, the equilibrium concept is refined by using the Intuitive Criterion of Cho and Kreps (1987).[15] Denote U_h^b the surplus of an h-type buyer and U_ℓ^b the surplus of an ℓ-type buyer in a proposed equilibrium of the bargaining game. This proposed equilibrium fails the Intuitive Criterion if there is an out-of-equilibrium offer $(\tilde{y}, \tilde{d}, \tilde{\tau}) \in \mathcal{F}$ and a buyer's type $\chi \in \{\ell, h\}$ such that the following is true:

$$u(\tilde{y}) - \kappa_\chi \tilde{d} - \tilde{\tau} \quad > \quad U_\chi^b \tag{14}$$

$$u(\tilde{y}) - \kappa_{-\chi} \tilde{d} - \tilde{\tau} \quad < \quad U_{-\chi}^b \tag{15}$$

$$-c(\tilde{y}) + \kappa_\chi \tilde{d} + \tilde{\tau} \quad \geq \quad 0, \tag{16}$$

where $\{-\chi\} = \{\ell, h\} \backslash \{\chi\}$. According to (14), the offer $(\tilde{y}, \tilde{d}, \tilde{\tau})$ would make a χ-type buyer strictly better off if it were accepted. According to (15), the offer $(\tilde{y}, \tilde{d}, \tilde{\tau})$ would make the $-\chi$-type buyer strictly worse off. According to (16), the offer is acceptable provided that the seller believes it comes from a χ-type.

[14] A similar tie-breaking assumption is used in Rubinstein (1985, Assumption B-3). It is made so that the set of acceptable offers is closed, and the buyer's problem has a solution.

[15] The Intuitive Criterion is a refinement supported by much of the signalling literature. An equilibrium that fails the Intuitive Criterion gives an outcome that is not strategically stable in the sense of Kohlberg and Mertens (1986). See Riley (2001) for a survey of the applications of the Intuitive Criterion (and other refinements) in various contexts. It has been used in monetary theory by Nosal and Wallace (2007); in the corporate finance literature by Noe (1989) and DeMarzo and Duffie (1999); in bargaining theory by Rubinstein (1985, Assumption B-1); and, recently, in the literature on global games by Angeletos, Hellwig, and Pavan (2006). In our context, the Intuitive Criterion has the additional advantage of preserving the tractability of the model once the bargaining game is embodied in the general equilibrium structure in Section 5. For the sake of completeness, the model is also analyzed under the alternative refinement from Mailath, Okuno-Fujiwara, and Postlewaite (1993) in the Appendix C of Rocheteau (2009a). See Footnote 19.

Definition 1 *An equilibrium of the bargaining game is a pair of strategies and a belief system,* $\langle [y(\kappa),d(\kappa),\tau(\kappa)],\ \mathcal{A}, \lambda\rangle$, *such that: (i)* $[y(\kappa), d(\kappa), \tau(\kappa)]$ *is solution to (13) with* $\kappa \in \{\kappa_\ell, \kappa_h\}$*; (ii)* $\mathcal{A}$ *is given by (12); (iii)* $\lambda : \mathcal{F} \to [0,1]$ *satisfies Bayes's rule whenever possible and the Intuitive Criterion.*

The next Lemma narrows the set of possible equilibria.

Lemma 1 *In equilibrium, there is no pooling offer with* $d \neq 0$.

Any equilibrium in which there are transfers of equity between buyers and sellers is separating. The logic of the argument goes as follows. Suppose there is a pooling offer such that $d > 0$. The buyer in the high-dividend state has the possibility of signaling the quality of the equity by choosing an offer that would raise his payoff relative to the proposed equilibrium, but such an offer would hurt buyers in the low-dividend state. The trade involves a lower level of consumption and a smaller transfer of equity compared to the equilibrium offer, but also a better price for the asset. (See Section 4.1 for a graphical illustration of this argument.) Reciprocally, if $d < 0$, then the buyer in the low-dividend state can reduce his transfer of bonds and the quantity of equity he buys from the seller in order to signal the low quality of the asset.

Lemma 2 *Any offer made by a buyer in the low-dividend state is such that*

$$(y_\ell, d_\ell, \tau_\ell) \in \arg\max_{y,\tau,d} [u(y) - \kappa_\ell d - \tau] \tag{17}$$

$$s.t. \quad -c(y) + \kappa_\ell d + \tau \geq 0 \tag{18}$$

$$-z^s \leq \tau \leq z^b, \quad 0 \leq d \leq a^b. \tag{19}$$

Any offer made by a buyer in the high-dividend state is such that

$$(y_h, d_h, \tau_h) \in \arg\max_{y,\tau,d} [u(y) - \kappa_h d - \tau] \tag{20}$$

$$s.t. \quad -c(y) + \kappa_h d + \tau \geq 0 \tag{21}$$

$$u(y) - \kappa_\ell d - \tau \leq u(y_\ell) - c(y_\ell) \tag{22}$$

$$-z^s \leq \tau \leq z^b, \quad -a^s \leq d \leq a^b. \tag{23}$$

The only way an ℓ-type buyer can achieve a higher payoff than the one he would get in a game with complete information is by making an offer with $d_\ell > 0$ that a seller would attribute to an h-type buyer with positive probability, but this has been ruled out by Lemma 1. Hence, buyers in the low-dividend state make their complete information offer (which is always acceptable, provided that $d_\ell \geq 0$, irrespective of sellers' beliefs). The solution to (17)-(19) is

$$y_\ell = y^*,$$
$$\kappa_\ell d_\ell + \tau_\ell = c(y^*),$$

if $\kappa_\ell a^b + z^b \geq c(y^*)$. Notice from (19) that the buyer in the low state is restricted from buying equity from the seller, $d_\ell \geq 0$. This guarantees that buyers in the high state have no incentives to mimic the offer made by buyers in the low state. If $\kappa_\ell a^b + z^b < c(y^*)$, then

$$\tau_\ell = z^b,$$
$$d_\ell = a^b,$$
$$y_\ell = c^{-1}\left(\kappa_\ell a^b + z^b\right).$$

According to the Intuitive Criterion, an h-type buyer can always increase his payoff as long as by so doing he does not give incentives to an ℓ-type buyer to imitate him. Hence, from (20)-(23), the buyer maximizes his surplus, subject to the participation constraint of the seller, where the seller has the correct belief that he faces an h-type buyer, and subject to the incentive-compatibility condition according to which an ℓ-type buyer does not want to mimic the offer of an h-type buyer.

A belief system consistent with the offers in Lemma 2 is such that sellers attribute all offers that would raise the payoff of buyers in the low-dividend state relative to their complete information payoff to ℓ-type buyers, and all other out-of-equilibrium offers to h-type buyers. Offers that violate (22) also violate (18) and, since they are attributed to ℓ-type buyers, they are rejected.

Proposition 1 ***(A pecking order theory of payments)***

Consider a match between a buyer holding a portfolio (z^b, a^b) and a seller holding a portfolio (z^s, a^s). There is a solution (y_h, d_h, τ_h) to (20)-(23), and it has the following properties:

1. *If $z^b \geq c(y^*)$, then*

$$y_h = y^* \tag{24}$$
$$\tau_h + \kappa_h d_h = c(y^*) \tag{25}$$
$$d_h \leq 0. \tag{26}$$

2. *If $z^b < c(y^*)$, then $\tau_h = z^b$ and $(y_h, d_h) \in [0, y_\ell] \times [0, a^b]$ is the unique solution to:*

$$\kappa_h d_h = c(y_h) - z^b \quad (27)$$

$$u(y_h) - c(y_h) + \left(1 - \frac{\kappa_\ell}{\kappa_h}\right)\left[c(y_h) - z^b\right] = u(y_\ell) - c(y_\ell), \quad (28)$$

where $y_\ell = \min\left[y^*, c^{-1}\left(z^b + \kappa_\ell a^b\right)\right]$. Moreover, if $a^b > 0$, then $y_h < y_\ell$ and $d_h \in \left(0, a^b\right)$.

Proposition 1 offers a *pecking order* theory of payment choices: agents with a consumption opportunity finance it with risk-free bonds first, and they use their risky assets as a last resort.[16] If buyers hold enough bonds to buy y^* ($z^b \geq c(y^*)$), then they do not transfer any equity to the sellers. In this sense, the risk-free bond is a preferred means of payment. Even when buyers do not have enough wealth to buy the surplus-maximizing level of output, they choose not to spend all their equity. By retaining a fraction of their equity holdings, buyers signal the high future dividend of the asset, and hence they secure better terms of trade.[17]

The fraction $\theta_h \equiv d_h/a^b$ of the equity holdings that a buyer spends in the DM is a function of his portfolio and the characteristics of the dividend process. For the functional forms $u(y) = 2\sqrt{y}$ and $c(y) = y$ the closed-form solution for θ_h is

$$\theta_h\left(\kappa_h, z^b, a^b\right) = \frac{\left(\frac{\kappa_h}{\kappa_\ell}\right)^2\left[1 - \sqrt{1 - \frac{\kappa_\ell}{\kappa_h}\left[2\sqrt{y_\ell} - y_\ell + \left(1 - \frac{\kappa_\ell}{\kappa_h}\right) z^b\right]}\right]^2 - z^b}{\kappa_h a^b} \quad \text{if } z^b \leq y^*$$
$$= 0 \text{ otherwise,}$$

where $y_\ell = \min\left(1, \kappa_\ell a^b + z^b\right)$ and $\kappa_\ell = \left(1 - \pi_h \kappa_h\right)/\pi_\ell$. This expression points to the differences between the approach in this paper and the approaches of Kiyotaki and Moore (2005) and Lagos (2006). In Kiyotaki and Moore (2005), agents can only sell a fraction, $\theta \in (0, 1)$, of their illiquid asset (capital) to raise funds; in Lagos (2006), agents can use their illiquid asset ("Lucas trees") in a fraction, θ, of the matches. In both cases, the parameter θ is exogenous. In contrast, in my model buyers spend a fraction, θ_h, of their equity shares when the dividend is high, where θ_h is a function of the intrinsic characteristics of the asset (κ_ℓ and

[16] The term "pecking order" was coined by Myers (1984, p.581). It describes the predictions of models of capital structure choices under private information. According to the pecking order theory, firms with an investment opportunity prefer internal finance (nondistributed dividends). If external finance is required, then they issue the safest security first, and they use equity as a last resort.

[17] This result is reminiscent of some of the findings of the liquidity-based model of security design from DeMarzo and Duffie (1999). They consider the problem faced by a firm that needs to raise funds by issuing a security backed by real assets. The issuer has private information regarding the distribution of cash flows of the underlying assets. Using the Intuitive Criterion, they show that a signaling equilibrium exists in which the seller receives a high price for the security by retaining some fraction of the issue.

κ_h) and the composition of the portfolio held by the buyer (z^b and a^b). Hence, the (il)liquidity of the equity depends on its intrinsic characteristics, as well as policy, which determines the supply of the risk-free asset. Finally, my approach provides guidelines on how to formalize trading restrictions: are buyers restricted from using equity in a fraction of the meetings, or required to use only a fraction of their equity holdings in all meetings? My model suggests a combination of the two restrictions: buyers retain a fraction of their equity holdings in meetings when the dividend state is high.

Proposition 2 ***(Asset liquidity and fundamentals)***

Assume $z^b < c(y^*)$ and $a^b > 0$. Then:

1. $\frac{\partial \theta_h}{\partial \kappa_h} < 0$.

2. $\lim_{\kappa_\ell \to 0} \theta_h = 0$.

Part 1 of Proposition 2 describes a mean-preserving increase in the spread of the distribution of dividends across states. The propensity to spend equity in the high-dividend state decreases as the distribution becomes riskier. To understand this result, notice from (22) that ℓ-type buyers enjoy an informational rent (the difference between the buyer's surplus in the low state and the buyer's surplus in the high state) equal to $(\kappa_h - \kappa_\ell)d_h > 0$. As κ_ℓ gets closer to κ_h, this informational rent shrinks, and the incentive-compatibility constraint is relaxed, which improves the liquidity of equity in the high-dividend state.[18] Conversely, as $\kappa_h - \kappa_\ell$ increases, the informational asymmetries become more severe, which makes the incentive-compatibility condition more binding. In the case where the dividend in the low state approaches 0, the adverse selection problem is so severe that equity ceases to be traded. Risk-free bonds become the only means of payment.[19]

Proposition 3 ***(Payments and portfolio composition)***

[18] This result is related to the findings in Banerjee and Maskin (1996), according to which the good that serves as the medium of exchange is the one for which the discrepancy between qualities is smallest.

[19] Strictly speaking, the ℓ-type buyers can still use equity shares in payments, but because κ_ℓ tends to 0, the amount of output they buy with it approaches 0. Also, a well-known property of the equilibrium selected by the Intuitive Criterion is that the outcome is independent of the distribution of types (π_h, π_ℓ), which can make the adverse-selection problem look very severe when the occurrence of the low state is infrequent. In Rocheteau (2009, Appendix C) I checked the robustness of the result to the notion of undefeated equilibrium proposed by Mailath, Okuno-Fujiwara, and Postlewaite (1993). If $z^b \in [c(\hat{y}), c(y^*))$ where $\hat{y}$ is the solution to $u'(y) = \kappa_h c'(y)$, then the unique undefeated equilibrium corresponds to the one selected by the Intuitive Criterion.

1. If $z^b < c(y^*)$ and $a^b > 0$, then

$$\frac{\partial(\kappa_h d_h)}{\partial z^b} = \frac{u'(y_\ell)/c'(y_\ell) - u'(y_h)/c'(y_h)}{u'(y_h)/c'(y_h) - \frac{\kappa_\ell}{\kappa_h}} < 0. \tag{29}$$

2. If $\kappa_\ell a^b + z^b < c(y^*)$, then

$$\frac{\partial d_h}{\partial a^b} = \frac{u'(y_\ell)/c'(y_\ell) - 1}{\frac{\kappa_h}{\kappa_\ell} u'(y_h)/c'(y_h) - 1} \in (0, 1). \tag{30}$$

As the buyer's bonds holdings increase, the transfer of equity (expressed in CM output) decreases. The buyer uses his additional bonds to reduce d_h, thereby relaxing the incentive-compatibility constraint (22). This dependence of θ_h on z^b will offer a channel through which policy affects the liquidity of equity.

According to (30), the marginal propensity of a buyer to spend his equity in the high-dividend state is less than one. Provided that $\kappa_\ell a^b + z^b < c(y^*)$, an additional unit of equity increases the surplus of the buyer in the low-dividend state, and hence it relaxes the incentive-compatibility constraint in the high-dividend state, which allows the buyer to spend a fraction of his marginal unit of equity. If $\kappa_\ell a^b + z^b > c(y^*)$, then $y_\ell = y^*$ and $\partial d_h / \partial a^b = 0$. In this case, the liquidity needs in the low-dividend state are satiated and, as a result, an additional unit of equity does not affect the incentive-compatibility constraint, and hence the terms of trade, in the high-dividend state.

4.1 A benchmark

In the following, I describe the economy in which there is no intervention by the government to supply risk-free bonds, $z^s = z^b = 0$. This special case provides a graphical illustration of the results in Lemma 1 and Proposition 1 and some insights on the role of risk-free assets.

As shown in Lemma 1, there is no equilibrium of the bargaining game with a pooling offer. The proof is illustrated in the left panel of Figure 2. Consider an equilibrium with a pooling offer $(\bar{y}, \bar{d})$ with $\bar{d} > 0$. (An offer with $\bar{d} < 0$ would not be acceptable since the seller would receive nothing in exchange for some output and some asset.) The surpluses of the two types of buyers at the proposed equilibrium are denoted $U_\ell^b \equiv u(\bar{y}) - \kappa_\ell \bar{d}$ and $U_h^b \equiv u(\bar{y}) - \kappa_h \bar{d}$. The indifference curves U_ℓ^b and U_h^b in Figure 2 represent the set of offers (y, d) that generate the equilibrium surpluses. They exhibit a single-crossing property, which is key to obtaining a separating equilibrium. The participation constraint of a seller who believes he is facing an h-type buyer is represented by the frontier $U_h^s \equiv \{(y, d) : -c(y) + \kappa_h d = 0\}$. The offer $(\bar{y}, \bar{d})$ is located above U_h^s since it is accepted when $\lambda < 1$. The shaded area indicates the set of offers that raise the utility of an h-type buyer (offers to the right of U_h^b), but reduce the utility of an ℓ-type buyer (offers to the left of U_ℓ^b), and are acceptable by sellers provided that $\lambda = 1$ (offers above U_h^s). These offers satisfy (14)-(16) with $\chi = h$ so that the proposed equilibrium with a pooling offer $(\bar{y}, \bar{d})$ violates the Intuitive Criterion. In order to separate himself, an h-type buyer reduces his DM consumption as well as his transfer of assets to the seller. Provided that the reduction in y is sufficiently large relative to the reduction in d, an ℓ-type buyer would not choose such an offer because his assets are less valuable than those of an h-type buyer.

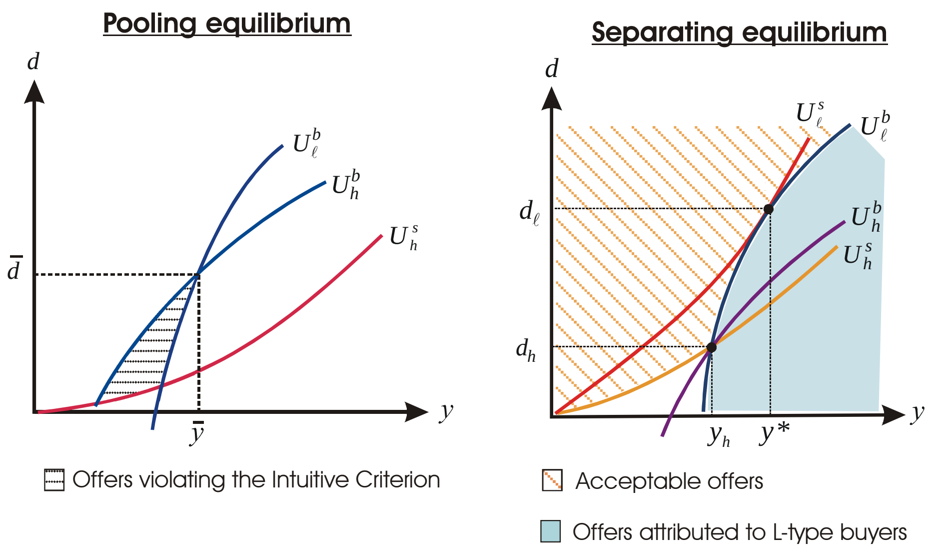

Figure 2: Pooling vs separating equilibria

The equilibrium offers are as described in Lemma 2. The buyer in the low-dividend state makes his complete-information offer while the buyer in the high-divided state makes the least-cost separating offer. Buyers' offers are illustrated in the right panel of Figure 2 in the case where the constraint $d_\ell \leq a^b$ does not bind. The offer of the ℓ-type buyer is at the tangency point between the iso-surplus curve of the seller, $U_\ell^s \equiv -c(y) + \kappa_\ell d = 0$, and the iso-surplus curve of the buyer, U_ℓ^b. In order to satisfy the seller's participation constraint and the incentive-compatibility condition, h-type buyers make offers to the left of U_ℓ^b and above U_h^s. The utility-maximizing offer is at the intersection of the two curves. As shown in the figure, and proved in Proposition 1, $y_h < y_\ell \leq y^*$. Buyers always consume less in the high-dividend state than in the low-dividend state. Despite this inefficiently low consumption, buyers retain a fraction of their equity holdings (Proposition 1).

Turn to the normative properties of the equilibrium. If $\kappa_\ell a^b \geq c(y^*)$, then the value of the low-dividend asset is large enough to trade the first-best quantity, y^*. Under complete information, the economy would achieve its first-best. In contrast, if the quality of the asset is private information, then the equilibrium allocation is inefficient. The ℓ-type buyers consume y^*, but h-type buyers consume $y_h < y^*$. If $\kappa_\ell a^b < c(y^*)$, then the quantities traded in the DM are inefficiently low in all matches, i.e., $y_h < y_\ell < y^*$.[20] The inefficiency induced by the private-information problem can be shown in a rather dramatic way by looking at the case in which κ_ℓ approaches 0, i.e., equity is valueless in one state. Then, from Proposition 2, θ_h goes to 0 so that buyers do not spend any of their real asset holdings, and the market shuts down ($y_h, y_\ell \to 0$).

As revealed by Proposition 1, by holding risk-free assets the buyer can overcome the inefficiency associated with the private-information problem. In particular, if $z^b \geq c(y^*)$, then the first-best allocation is obtained, and buyers do not use the real asset as means of payment in the high-dividend state.

5 Asset prices and liquidity

This section incorporates the bargaining game studied in Section 4 into the general equilibrium structure described in Section 2, and it investigates the implications of the model for the relationship between policy, assets' liquidity and returns. The objective is twofold. First, I will show qualitatively the potential of the model to account for standard asset-pricing puzzles, such as the risk-free rate and equity premium puzzles. I will establish conditions under which the risk-free rate is lower than the discount rate and the equity premium is positive despite quasilinear preferences. Second, I will show that a policy that consists of changing the supply of risk-free bonds has a permanent liquidity effect on the structure of asset returns, output, and welfare.

The sequence of events is as follows. (See Figure 1.) Agents make a portfolio choice in the CM. At the beginning of the subsequent period, buyers receive a private and fully informative signal about the future dividend of the equity, while sellers are uninformed. Then, buyers get matched with sellers. An implication of this timing is that the buyer's portfolio does not convey any information about his private information. From Proposition 1, the terms of trade $[y(z^b, a^b, \kappa), \tau(z^b, a^b, \kappa), d(z^b, a^b, \kappa)]$ are functions of the buyer's portfolio

[20] One could also ask whether there exists an incentive-feasible trading mechanism that implements the first-best allocation in the absence of risk-free bonds. Consider a direct mechanism that maps the buyer's type, κ, into an offer (y, d). Suppose $y_h = y_\ell = y^*$. Then, incentive-compatibility constraint requires $d_h = d_\ell = d$. So the outcome is pooling. The trade (y^*, d) satisfies the seller's individual rationality constraint if $-c(y^*) + d \geq 0$. Similarly, buyers are willing to participate if $u(y^*) - \kappa_h d \geq 0$. Thus, the first-best is incentive-feasible provided that $a^b \geq c(y^*)$ and $\kappa_h \leq u(y^*)/c(y^*)$, i.e., there is no shortage of the asset, and the discrepancy between the dividends in the different states is not too large.

and his private signal.[21]

The missing element of the model so far is the determination of agents' portfolio choices. These choices depend on the benefits that an agent expects to receive from holding assets in the DM. The expected lifetime utility of a buyer entering the DM with z units of bonds, a units of equity, and a private signal κ, is

$$V^b(z,a,\kappa) = u\left[y(z,a,\kappa)\right] + W^b\left[z-\tau(z,a,\kappa), a-d(z,a,\kappa), \kappa\right]. \tag{31}$$

Using the linearity of W^b, (31) becomes

$$V^b(z,a,\kappa_\chi) = S^\chi(z,a) + z + \kappa_\chi a + W^b(0,0,\kappa_\chi), \quad \chi\in\{\ell,h\}, \tag{32}$$

where $S^\chi(z,a)$ is the buyer's surplus in the DM when the dividend state is χ, i.e.,

$$S^\chi(z,a) \equiv u\left[y(z,a,\kappa_\chi)\right] - \kappa_\chi d(z,a,\kappa_\chi) - \tau(z,a,\kappa_\chi) \text{ for } \chi\in\{\ell,h\}.$$

Substituting V^b by its expression given by (32) into (10), the buyer's portfolio problem reduces to

$$[z(j),a(j)] \in \arg\max_{(z,a)\in\mathbb{R}_{2+}} \left\{-\left(\frac{q_z-\beta}{\beta}\right)z - \left(\frac{q_a-\beta}{\beta}\right)a + \pi_h S^h(z,a) + \pi_\ell S^\ell(z,a)\right\}, \quad \forall j\in\mathcal{B}, \tag{33}$$

where $\frac{q_z}{\beta}-1$ is the cost of holding bonds, and $\frac{q_a}{\beta}-1$ is the cost of investing in equity. The cost of holding an asset is approximately equal to the difference between the price of the asset and its fundamental value, β. According to (33), buyers choose their portfolios in order to maximize their expected surplus in the DM, net of the cost of holding the assets. Since sellers obtain no surplus from their DM trades, their expected lifetime utility upon entering the DM is

$$V^s(z,a) = \mathbb{E}W^s(z,a,\kappa) = z + a + \mathbb{E}W^s(0,0,\kappa). \tag{34}$$

Substituting V^s from (34) into (11), the seller's choice of asset holdings is given by

$$[z(j),a(j)] \in \arg\max_{(z,a)\in\mathbb{R}_{2+}} \left\{-\left(\frac{q_z-\beta}{\beta}\right)z - \left(\frac{q_a-\beta}{\beta}\right)a\right\}, \quad \forall j\in\mathcal{S}. \tag{35}$$

Finally, the clearing of the asset market implies

$$\int_{j\in\mathcal{J}} a(j)dj = A. \tag{36}$$

$$\int_{j\in\mathcal{J}} z(j)dj = Z. \tag{37}$$

[21]The solutions to (17)-(19) and (20)-(23) might not be unique, e.g., if $z^b > c(y^*)$, but agents' surpluses are unique, and they are independent of the seller's asset holdings.

Definition 2 *An equilibrium is a list of portfolios, terms of trade in the DM and the prices of equity and bonds,* $\langle [z(j), a(j)]_{j\in\mathcal{J}}, [y(\cdot), d(\cdot), \tau(\cdot)], q_a, q_z\rangle$ *such that:*

(i) $[z(j), a(j)]$ *is solution to (33) for all* $j \in \mathcal{B}$ *and to (35) for all* $j \in \mathcal{S}$*;*

(ii) For all $(z, a) \in \mathbb{R}_{2+}$*,* $[y(z, a, \kappa), d(z, a, \kappa), \tau(z, a, \kappa)]$ *is a solution to (17)-(19) if* $\kappa = \kappa_\ell$ *and to (20)-(23) if* $\kappa = \kappa_h$*;*

(iii) (q_a, q_z) *solves (36) and (37).*

The next two lemmas characterize the buyers' and sellers' portfolio choices.

Lemma 3 ***(Sellers' portfolio choices)***

Consider the seller's portfolio problem in (35). It has a solution if and only if $q_z \geq \beta$ *and* $q_a \geq \beta$*.*

1. *If* $q_z > \beta$*, then* $z = 0$*. If* $q_z = \beta$*, then* $z \in [0, \infty)$*.*

2. *If* $q_a > \beta$*, then* $a = 0$*. If* $q_a = \beta$*, then* $a \in [0, \infty)$*.*

The proof, which is immediate from (35), is omitted. Sellers, who get no surplus from trades in the DM, hold an asset only if its price is equal to its fundamental value. Let S_z^χ and S_a^χ denote the partial derivatives of the buyer's surplus function for $\chi \in \{\ell, h\}$. These quantities represent the transactional benefits to a buyer that bonds and equity provide at the margin in the DM in the dividend state χ.

Lemma 4 ***(Buyers' portfolio choices)***

If $q_z \geq \beta$ *and* $q_a \geq \beta$*, then* (z, a) *is a solution to the buyer's portfolio problem, (33), if and only if*

$$-\frac{q_z - \beta}{\beta} + \pi_h S_z^h(z, a) + \pi_\ell S_z^\ell(z, a) \leq 0 \quad \text{“} = \text{”} \quad \text{if } z > 0, \tag{38}$$

$$-\frac{q_a - \beta}{\beta} + \pi_h S_a^h(z, a) + \pi_\ell S_a^\ell(z, a) \leq 0 \quad \text{“} = \text{”} \quad \text{if } a > 0, \tag{39}$$

where

$$S_z^\ell = \frac{S_a^\ell}{\kappa_\ell} = \frac{u'(y_\ell)}{c'(y_\ell)} - 1 \tag{40}$$

$$S_z^h = \left[\frac{u'(y_h)}{c'(y_h)} - 1\right]\left[\frac{u'(y_\ell)/c'(y_\ell) - \kappa_\ell/\kappa_h}{u'(y_h)/c'(y_h) - \kappa_\ell/\kappa_h}\right] \tag{41}$$

$$S_a^h = \kappa_h\left[\frac{u'(y_h)}{c'(y_h)} - 1\right]\frac{\kappa_\ell}{\kappa_h}\left[\frac{u'(y_\ell)/c'(y_\ell) - 1}{u'(y_h)/c'(y_h) - \frac{\kappa_\ell}{\kappa_h}}\right]. \tag{42}$$

If $q_z > \beta$ and $q_a > \beta$, then (z, a) is unique.

If $q_z = \beta$, then $z \geq c(y^*)$.

If $q_a = \beta$, then $z + \kappa_\ell a \geq c(y^*)$.

From (38) and (39), for an asset to be held by buyers, its cost must be equal to the expected marginal benefit that the asset confers to the buyers in the DM. According to (40), a marginal unit of an asset allows the buyer to purchase $1/c'(y_\ell)$ units of DM output when the dividend state is low; this additional output is valued according to the marginal surplus of the match, $u'(y_\ell) - c'(y_\ell)$. The first term in brackets on the right side of (41) is the liquidity value of bonds in the high-dividend state in the complete-information economy. This term is multiplied by $1 + \frac{\partial(\kappa_h d_h)}{\partial z^b} < 1$ because, in the private-information economy, the buyer with an additional unit of bond reduces his transfer of equity in order to mitigate the informational asymmetry in the match. Similarly, the first two terms on the right side of (42) correspond to the liquidity value of equity in the high-dividend state in the complete-information economy. This liquidity component is multiplied by the marginal propensity to spend the equity, $\frac{\partial d_h}{\partial a^b} \in [0, 1)$, which is less than one in the private-information economy.

If asset prices are greater than their fundamental values, i.e., $q_a > \beta$ and $q_z > \beta$, then the buyer's optimal portfolio is unique. This result is a consequence of Proposition 1, according to which bonds and equity are imperfect substitutes. Since the two assets do not perform the same role–risk-free bonds are preferred means of payment–there is an optimal composition of the buyer's portfolio. If equity is priced according to its fundamental value, $q_a = \beta$, then the buyer's choice of equity is indeterminate: buyers accumulate enough wealth to buy the first-best quantity of output when $\kappa = \kappa_\ell$. If the rate of return of bonds is equal to the discount rate, i.e., $q_z = \beta$, then buyers accumulate enough bonds to buy the first-best level of output in all states.

Proposition 4 ***(Equilibrium allocations and prices)***

An equilibrium exists, and it is such that (q_a, q_z, y_ℓ, y_h) is uniquely determined. Asset prices are

$$q_z = \beta (1 + \mathcal{L}_z) \tag{43}$$

$$q_a = \beta(1 + \mathcal{L}_a), \tag{44}$$

with

$$\mathcal{L}_z = \pi_\ell \left[\frac{u'(y_\ell)}{c'(y_\ell)} - 1\right] + \pi_h \left[\frac{u'(y_h)}{c'(y_h)} - 1\right] \left[\frac{\kappa_h u'(y_\ell)/c'(y_\ell) - \kappa_\ell}{\kappa_h u'(y_h)/c'(y_h) - \kappa_\ell}\right], \tag{45}$$

$$\mathcal{L}_a = \pi_\ell \kappa_\ell \left[\frac{u'(y_\ell)}{c'(y_\ell)} - 1\right] + \pi_h \kappa_h \left[\frac{u'(y_h)}{c'(y_h)} - 1\right] \left[\frac{\kappa_\ell u'(y_\ell)/c'(y_\ell) - \kappa_\ell}{\kappa_h u'(y_h)/c'(y_h) - \kappa_\ell}\right], \tag{46}$$

where $y_\ell = \min\left[y^*, c^{-1}(Z + \kappa_\ell A)\right]$ *and* y_h *solves (28), with* $z = Z$.

An equilibrium exists, and it is essentially unique.[22] The asset prices, q_z and q_a, are determined from (38) and (39) at equality, where $a = A$ and $z = Z$. The price of each asset is composed of its fundamental value, β, times a liquidity factor. The liquidity factors, $\mathcal{L}_z$ and $\mathcal{L}_a$, coincide with their expressions in the complete-information economy, except for the last terms in brackets on the right sides of (45) and (46).[23] The term $\frac{\kappa_h u'(y_\ell)/c'(y_\ell) - \kappa_\ell}{\kappa_h u'(y_h)/c'(y_h) - \kappa_\ell} = 1 + \kappa_h \frac{\partial d_h}{\partial z^b}$ captures the fact that when the buyer accumulates additional units of the risk-free asset, he reduces the quantity of equity he transfers to the seller $\left(\frac{\partial d_h}{\partial z^b} < 0\right)$. Similarly, the private information friction reduces the liquidity value of equity, $\frac{\kappa_\ell u'(y_\ell)/c'(y_\ell) - \kappa_\ell}{\kappa_h u'(y_h)/c'(y_h) - \kappa_\ell} = \frac{\partial d_h}{\partial a_b} < 1$, because the buyer's marginal propensity to spend equity is less than one.

The next Proposition determines the condition under which asset prices depart from their fundamental values, i.e., when the assets' liquidity components, $\mathcal{L}_z$ and $\mathcal{L}_a$, are positive. Let $r_z \equiv \frac{1}{q_z}$ denote the (gross) risk-free rate and $r_a \equiv \frac{1}{q_a}$ the (gross) rate of return of equity.

Proposition 5 ***(Liquidity and asset returns)***

1. $\mathcal{L}_z = 0$ *and* $r_z = \beta^{-1}$ *if and only if* $Z \geq Z^* \equiv c(y^*)$.

2. $\mathcal{L}_a = 0$ *and* $r_a = \beta^{-1}$ *if and only if* $Z \geq \bar{Z} \equiv c(y^*) - \kappa_\ell A$.

3. *If* $Z < Z^*$, *then* $\mathcal{L}_z > \mathcal{L}_a \geq 0$ *and* $\beta^{-1} \geq r_a > r_z$.

The risk-free rate is below the rate of time preference if the quantity of risk-free assets in the economy is not large enough to allow buyers to consume the first-best level of output in the DM. In this case, bonds have a positive liquidity value irrespective of the economy-wide stock of equity.[24] Bonds are useful, even for large

[22] Any indeterminacy, such as the composition of the payments in terms of equity and bonds in the low-dividend state when $Z + \kappa_\ell A > c(y^*)$, is payoff irrelevant.

[23] See Appendix D in Rocheteau (2009a) for a derivation of the asset prices in the complete-information economy.

[24] Huggett (1993) provides a related explanation for the low risk-free interest rate. He considers an economy in which agents experience uninsurable idiosyncratic shocks and partially insure themselves by holding a risk-free asset.

values of A, because they overcome the partial illiquidity of the equity in the high-dividend state, i.e., they relax the incentive-compatibility constraint faced by buyers, which allows them to consume more. Equity shares can also be valued for their liquidity services if economy-wide wealth in the low-dividend state is too scarce to allow the implementation of the first-best allocation, $Z + \kappa_\ell A < c(y^*)$.[25] The liquidity value of bonds, however, is greater than the liquidity value of equity. As a consequence, the rate-of-return differential between bonds and equity, the equity premium, is positive.

Since agents are risk-neutral with respect to CM consumption, the equity premium does not emerge from the standard risk-aversion component of the pricing kernel. Risk matters here for two reasons. First, the riskiness of equity generates a covariance between its dividend and the marginal value of wealth in the DM. This covariance is negative in the complete-information economy because a high dividend is associated with a high wealth. This reduces the liquidity value of equity relative to bonds.[26] In contrast, in the private-information economy, the DM output is lower in the high-dividend state relative to the low-dividend state. (See Proposition 1.) Second, the riskiness of equity makes the informational asymmetry between buyers and sellers relevant. It is because the dividend of equity can take different values, and because buyers have some private information about the future value of the asset, that risk-free bonds are preferred means of payment.

Figure 3 represents the conditions on Z and A under which the liquidity factors $\mathcal{L}_z$ and $\mathcal{L}_a$ are positive for both the complete-information and the private-information economies. In the complete-information economy (right panel of Figure 3) either all asset prices exhibit a liquidity component, or none of them does. Provided that there is enough wealth in the economy, $\kappa_\ell A + Z \geq c(y^*)$, assets are priced according to their fundamental values. In contrast, in the private-information economy (left panel of Figure 3) bonds can be priced above their fundamental value even if the total wealth in the economy is very large. Consequently, a liquidity differential between equity and bonds (the grey areas in Figure 3) is more likely to exist in the presence of an informational asymmetry.

How do asset prices compare in the private-information and the complete-information economies? Asset

[25] Weill (2008) also relates the liquidity of an asset to its supply. In his model, due to search frictions and increasing returns in the matching process, an asset with more tradeable shares is easier to find.

[26] This finding explains why a rate-of-return dominance pattern can emerge from a complete-information economy. See Rocheteau (2009a, Appendix D).

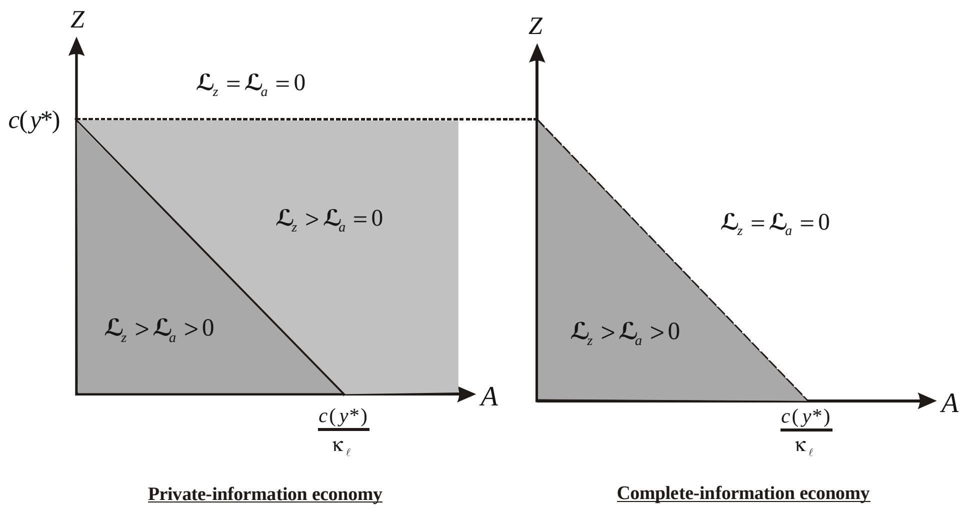

Figure 3: Liquidity and asset prices

prices in the complete-information economy are

$$\frac{q_z^{ci}}{\beta} = 1 + \pi_\ell \left[\frac{u'(y_\ell^{ci})}{c'(y_\ell^{ci})} - 1\right] + \pi_h \left[\frac{u'(y_h^{ci})}{c'(y_h^{ci})} - 1\right] \tag{47}$$

$$\frac{q_a^{ci}}{\beta} = 1 + \pi_\ell \kappa_\ell \left[\frac{u'(y_\ell^{ci})}{c'(y_\ell^{ci})} - 1\right] + \pi_h \kappa_h \left[\frac{u'(y_h^{ci})}{c'(y_h^{ci})} - 1\right], \tag{48}$$

with $y_\ell^{ci} = \min\left[y^*, c^{-1}(Z + \kappa_\ell A)\right] \leq y_h^{ci} = \min\left[y^*, c^{-1}(Z + \kappa_h A)\right]$.

Proposition 6 ***(Private information and asset prices)***

If $Z < Z^*$, then $q_z > q_z^{ci}$ and $q_z - q_a > q_z^{ci} - q_a^{ci}$.

If $Z < c(y^*)$, then the price of bonds is greater, and the risk-free rate lower, than their counterparts in the complete-information economy. Moreover, the price difference between bonds and equity is larger in the presence of an informational asymmetry. These results have empirical implications for asset-pricing anomalies. Lagos (2006) calibrates a related search model of exchange with complete information and shows that the model generates a risk-free rate which is too high and an equity premium which is too low (for plausible degrees of risk aversion). Under a mild restriction–in 2 percent of the DM trades buyers cannot use equity to finance consumption opportunities–the sizes of the risk-free rate and the equity premium can be made comparable to the ones in data. Proposition 6 suggests that such restrictions might not be needed in economies where informational asymmetries are prevalent.

The next Proposition specifies how the discrepancy between the dividend in different states affects the structure of asset returns.

Proposition 7 ***(Safety and rates of return)***

1. *If* $\kappa_\ell/\kappa_h = 1$*, then* $r_a = r_z$ *for all* $Z \geq 0$*.*

2. *If* $\kappa_\ell/\kappa_h \to 0$*, then* $r_a \to \beta^{-1} \geq r_z$*, with a strict inequality if* $Z < Z^*$*.*

If equity is safe, then bonds and equity have the same rate of return. If equity is so risky that it is valueless in the low state, then the turnover of the equity in the high-dividend state falls to 0, and its rate of return approaches its maximum given by the discount rate.

I now turn to the effects of policy on asset returns. Policy is described as the change in the supply of risk-free bonds.

Proposition 8 ***(Policy and asset prices)***

1. *If* $Z < Z^*$*, then* $d\mathcal{L}_z/dZ < 0$ *and* $dr_z/dZ > 0$*.*

2. *If* $Z < \bar{Z}$*, then* $d\mathcal{L}_a/dZ < 0$ *and* $dr_a/dZ > 0$*.*

3. *If* $Z < Z^*$*, then* $\frac{\partial(\mathcal{L}_z - \mathcal{L}_a)}{\partial Z} < 0$ *and* $\frac{\partial\left[\frac{r_a - r_z}{r_z r_a}\right]}{\partial Z} < 0$*.*

4. *For all* $Z \geq Z^*$*,* $y_\ell = y_h = y^*$ *and* $r_z = r_a = \beta^{-1}$*.*

If the risk-free rate is below the rate of time preference, then an increase in the supply of bonds raises the risk-free rate. The output in the DM, and hence social welfare, increase as well.[27] The liquidity effect associated with a change in the supply of bonds is permanent, and it prevails for any quantity of equity in the economy. Policy also affects the rate of return of equity and the equity premium if there is a shortage of wealth in the low state. An increase in the supply of bonds raises the return of equity, and it reduces the rate-of-return differential between assets.

[27] If the stock of risky capital was endogenous, then an increase in the supply of risk-free assets could also lead to a reduction in the capital stock.

The "Ricardian equivalence" fails to hold because there is a social role for the provision of risk-free assets.[28] The optimal policy consists of supplying enough bonds so as to satiate the demand for safe assets. As Z tends to Z^*, then y_ℓ and y_h approach y^*. In the high-dividend state, buyers trade only bonds ($d_h \to 0$), while in the low-dividend state buyers are indifferent between using bonds or equity to finance their consumption opportunities. The prices of bonds and equity converge to their fundamental values ($q_a = \beta$ and $q_z = \beta$).

6 Introducing fiat money

In Rocheteau (2008, 2009a), I show that fiat money can perform the same safety role as government bonds. Consider the case where bonds are replaced by fiat money, and the supply of money grows at the constant rate $\gamma > \beta$. At a stationary monetary equilibrium, the rate of return of fiat money is γ^{-1}. Let $i \equiv \frac{\gamma}{\beta} - 1$ denote the cost of holding real balances.

Proposition 9 ***(Equilibrium with fiat money)***

1. *For all $A > 0$, there is a $i_0(A) > 0$ such that the equilibrium is monetary if and only if $i < i_0(A)$.*

2. *If $i < i_0(A)$, then $r_a > \gamma^{-1}$.*

3. *For all $i > 0$, there is $\bar{A}(i) \in [0, c(y^*)/\kappa_\ell]$ such that for all $A < \bar{A}(i)$, $\mathcal{L}_a > 0$ and $r_a < \beta^{-1}$.*

4. *If $i < i_0(A)$ and $A < \bar{A}(i)$, then $d\mathcal{L}_a/di > 0$ and $dr_a/di < 0$.*

This proposition is proved in Rocheteau (2009a). A monetary equilibrium exists for all A provided that the cost of holding money, i, is not too large. The expected rate of return of equity is then larger than the rate of return of currency. Moreover, the price of equity exhibits a liquidity component if its stock is not too large and inflation is in some intermediate range. By reducing the rate of return of fiat money, an increase in γ is analogous to a reduction in the quantity of risk-free assets. Hence, an increase in inflation raises the price of equity, and it reduces its rate of return. The optimal monetary policy is to the Friedman rule, which

[28] Woodford (1990) promotes a related model as an explanation of the nonneutrality of government deficits. He considers an economy with alternating endowments where agents are liquidity-constrained in that they are unable to borrow against their future income at a rate of interest as low as that at which the government borrows. In his words, "A higher public debt, insofar as it implies a higher proportion of liquid assets in private sector wealth, increases the flexibility of the private sector in responding to variations in both income and spending opportunities, and so can increase economic efficiency."

requires $\gamma \to \beta$. As the rate of return of currency tends to the discount rate, the quantity of real balances approaches Z^* and output levels in the DM approach their first-best.

In the rest of the section, I propose an extension of the model with both fiat money and interest-bearing bonds. Without additional frictions, bonds and fiat money are perfect substitutes, and they must offer the same rate of return. Fiat money is valued if $Z < Z^*$ and $i < \mathcal{L}_z$, where $\mathcal{L}_z$ is evaluated at the nonmonetary equilibrium. In order to distinguish bonds from fiat money, I follow Freeman (1985) and Lester, Postlewaite, and Wright (2007) and assume that all assets except fiat money can be counterfeited at no cost. Moreover, only a fraction θ of the sellers are able to authenticate bonds and equity shares. The remaining quantity, $1-\theta$, of sellers cannot distinguish genuine bonds and equity shares from counterfeits, and hence they accept only fiat money in payment for goods.[29]

Let $\hat{y}$ denote the output level in the $1-\theta$ uninformed matches where sellers only accept fiat money. If m denotes the real balances of a buyer in a match, then $\hat{y}(m) = \min\left[y^*, c^{-1}(m)\right]$ and the buyer's surplus is $S^u(m) = u\left[\hat{y}(m)\right] - c\left[\hat{y}(m)\right]$. The buyer's portfolio problem becomes

$$(m, z, a) \in \arg\max \left\{ -im - \left(\frac{q_z - \beta}{\beta}\right) z - \left(\frac{q_a - \beta}{\beta}\right) a + (1-\theta)\, S^u(m) + \theta \left[\pi_h S^h(z+m, a) + \pi_\ell S^\ell(z+m, a)\right] \right\}.$$

Real balances affect both the buyer's surplus in uninformed matches and the buyer's expected surplus in informed matches. From the first-order condition with respect to m, the value of money solves

$$i = \theta \mathcal{L}_z + (1-\theta)\mathcal{L}_m, \tag{49}$$

with

$$\mathcal{L}_m = \frac{u'(\hat{y})}{c'(\hat{y})} - 1. \tag{50}$$

According to (49), the cost of holding fiat money must equal the sum of its liquidity returns in informed and uninformed matches. The liquidity value of money in uninformed matches corresponds to the increase in the buyer's surplus in the DM from buying $\frac{1}{c'(\hat{y})}$ unit of output with an additional unit of real balances.

[29] Arguably, the assumption that all assets except fiat money can be counterfeited at no cost is extreme. In Rocheteau (2009b) I extend Lester, Postlewaite, and Wright (2007) by assuming that claims on real assets can be counterfeited at a positive cost, and I show that in uninformed matches there is an endogenous upper bound on how much of their real asset holdings buyers can use as a means of payment. When the cost to produce counterfeits goes to zero, so does the upper bound on agents' capacity to spend their real assets.

Similarly, the rates of return on bonds and equity are determined by

$$\frac{q_z - \beta}{\beta} = \theta \mathcal{L}_z \tag{51}$$

$$\frac{q_a - \beta}{\beta} = \theta \mathcal{L}_a. \tag{52}$$

In contrast to fiat money, bonds and equity provide some liquidity services only in the fraction θ of the matches where they can be authenticated. Let $M = \int m(j)dj$ denote the aggregate real balances. An equilibrium can be reduced to a triple (M, q_z, q_a) that solves (49)-(52).

Proposition 10 ***(Monetary equilibrium)***

For all $\theta < 1$ and $\gamma > \beta$, there is a unique monetary equilibrium. There is $Z^{**} < Z^*$ such that for all $Z < Z^{**}$, $\gamma^{-1} < r_z < r_a \leq \beta^{-1}$.

There always exists a monetary equilibrium. Provided that the supply of bonds is not too large, the model predicts both a rate-of-return differential between currency and bonds and an equity premium. The next Proposition revisits the effects of policy in this environment. While a change in the nominal stock of money is neutral, a change in the supply of bonds can have real effects.

Proposition 11 ***(Policy)***

If $Z < Z^{**}$, then a reduction of Z leads to a reduction of r_z, a reduction of $M + Z$, and an increase of M.

Provided that Z is not too large, a change in the supply of bonds has a permanent liquidity effect that affects real interest rates, output, and welfare. A reduction of the stock of bonds leads to an increase in real balances and a decrease of the rate of return of bonds. The increase in real balances leads to an increase of the output in informed matches, $\hat{y}$. However, the real stock of money and bonds, $M + Z$, decreases, which generates a reduction of the output in uninformed matches, y_ℓ and y_h.

7 Conclusion

The objective of this paper was to provide a monetary theory of asset liquidity that emphasizes the role of assets in payment arrangements. The main ingredients of the model are the presence of multiple assets, risk-free bonds and risky equity, which are traded in both centralized and decentralized markets, and an informational asymmetry between agents paying with an asset and agents receiving the asset. I have explored the implications of the theory for the relationship between assets' intrinsic characteristics and liquidity, and the effects of policy on asset prices and welfare. To conclude, I review the empirical relevance of some of the predictions of the model.

A key finding of this paper is that the liquidity of an asset depends on the properties of its dividend process. An asset which is riskier tends to be less liquid. Several studies provide support for this finding. Krishnamurthy and Vissing-Jorgensen (2008) argue that half of the convenience yield of Treasury securities relative to corporate bonds can be explained by a *surety* motive, where *surety* is the "value investors place on a sure cash-flow above and beyond what would be implied by the pricing kernel." Spiegel and Wang (2005) find a strong negative correlation between liquidity and idiosyncratic risk in stock returns.

My model has asset-pricing implications that are consistent with the risk-free rate and equity premium puzzles. It predicts that the rate of return of government bonds is less than the rate of time preference, and risky equity commands a higher return than risk-free bonds despite agents being risk-neutral. While my results are qualitative, Lagos (2006) showed that a monetary model with a more standard pricing kernel and exogenous liquidity constraints could generate the observed risk-free rate and equity premium for plausible measures of risk aversion. Once fiat money is introduced, the model can generate both a rate-of-return difference between money and bonds, and an equity premium.

The model also predicts a negative relationship between the liquidity differential between risk-free bonds and risky assets and the supply of risk-free bonds. This is consistent with Krishnamurthy and Vissing-Jorgensen (2008), who find a negative relationship between the yield spread between corporate bonds and Treasury securities and the U.S. government debt-to-GDP ratio, based on annual observations from 1925 to 2005. Interestingly, this aggregate demand for Treasury debt is reminiscent of the aggregate money demand used by Lucas (2000) to assess the welfare cost of inflation. Lucas found that the welfare cost of 10 percent inflation is about 1 percent of GDP per year. Krishnamurthy and Vissing-Jorgensen found that "the value of the liquidity provided by the current level of Treasuries is around 0.95 per cent of GDP per year." Microfounded monetary models that have proven useful for assessing the welfare cost of inflation can also provide new insights for the social value of liquid assets.

In terms of monetary policy, Lagos and Rocheteau (2008) and Geromichalos, Licari, and Suarez-Lledo (2007) find that, in the absence of liquidity constraints, a monetary equilibrium ceases to exist if the supply of assets is sufficiently large. In contrast, my model has the more realistic prediction that a monetary equilibrium exists irrespective of the supply of assets, provided that the inflation rate is not too large. The rate of return of the asset is negatively correlated with inflation, in accordance with Danthine and Donaldson (1986) and Marshall (1992).

The natural next step is to construct a calibrated version of the model incorporating more realistic features, such as risk aversion, infinitely-lived assets, and a richer information structure. A more standard pricing kernel could be obtained by adopting the three-sector model of Lagos (2006), or the overlapping-generations model of Zhu (2008). My approach could also be useful for revisiting the relationship between inflation and capital, as in Aruoba, Waller, and Wright (2007).

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A. Proofs of lemmas and propositions

Proof of Lemma 1. The proof shows that any equilibrium with a pooling offer $(\bar{y}, \bar{d}, \bar{\tau})$ can be dismissed by the Intuitive Criterion. It distinguishes the case where $\bar{d} > 0$ from the case where $\bar{d} < 0$.

(i) Suppose first that $\bar{d} > 0$. By definition, the buyers' payoffs at the proposed equilibrium are $U_h^b = u(\bar{y}) - \kappa_h \bar{d} - \bar{\tau} \geq 0$ and $U_\ell^b = u(\bar{y}) - \kappa_\ell \bar{d} - \bar{\tau} \geq 0$. A necessary condition for $(\bar{y}, \bar{d}, \bar{\tau})$ to be acceptable when $\lambda(\bar{y}, \bar{d}, \bar{\tau}) < 1$ is $-c(\bar{y}) + \kappa_h \bar{d} + \bar{\tau} > 0$. Define $\mathcal{F} \equiv \mathbb{R}_+ \times [-a^s, a^b] \times [-z^s, z^b]$ the subspace of $\mathbb{R}^3$ of feasible offers. Let

$$O_1 \equiv \{(y, d, \tau) \in \mathcal{F} : -c(y) + \kappa_h d + \tau > 0\}.$$

Then, O_1 is open in $\mathcal{F}$, and it contains $(\bar{y}, \bar{d}, \bar{\tau})$. Let

$$O_2 \equiv \{(y, d, \tau) \in \mathcal{F} : \kappa_\ell \left(\bar{d} - d\right) < [u(\bar{y}) - \bar{\tau}] - [u(y) - \tau] < \kappa_h \left(\bar{d} - d\right)\}.$$

Then, O_2 is open in F, it is not empty, and its closure is $\overline{O}_2 \ni (\bar{y}, \bar{d}, \bar{\tau})$.[30] Consequently, any open ball centered at $(\bar{y}, \bar{d}, \bar{\tau})$ has a non-empty intersection with O_2. Moreover, by definition of an open set, there exists a radius $\varepsilon > 0$ such that the open ball $\mathcal{B}((\bar{y}, \bar{d}, \bar{\tau}), \varepsilon) \subset O_1$, and hence $\mathcal{B}((\bar{y}, \bar{d}, \bar{\tau}), \varepsilon) \cap O_2 \neq \varnothing$. Consequently, there is $(\tilde{y}, \tilde{d}, \tilde{\tau}) \in O_1 \cap O_2$ that satisfies (14)-(16) with $\chi = h$, and the proposed pooling equilibrium fails the Intuitive Criterion.

(ii) Suppose next that $\bar{d} < 0$. The participation constraint of the seller implies $\bar{\tau} > 0$. Since $\bar{d} < 0$ and $\lambda(\bar{y}, \bar{d}, \bar{\tau}) \in (0, 1)$ then

$$0 \leq -c(\bar{y}) + \lambda(\bar{y}, \bar{d}, \bar{\tau}) \kappa_h \bar{d} + \left[1 - \lambda(\bar{y}, \bar{d}, \bar{\tau})\right] \kappa_\ell \bar{d} + \bar{\tau} < -c(\bar{y}) + \kappa_\ell \bar{d} + \bar{\tau}.$$

Define

$$\begin{aligned} O_1 &\equiv \{(y, d, \tau) \in \mathcal{F} : -c(y) + \kappa_\ell d + \tau > 0\} \\ O_2 &\equiv \{(y, d, \tau) \in \mathcal{F} : \kappa_h \left(\bar{d} - d\right) < [u(\bar{y}) - \bar{\tau}] - [u(y) - \tau] < \kappa_\ell \left(\bar{d} - d\right)\}. \end{aligned}$$

In order to show that $O_2 \neq \varnothing$, one can construct an offer $(y, d, \tau) \in \mathcal{F}$ such that $y = \bar{y}$, $\bar{d} - d < 0$, $\tau - \bar{\tau} < 0$, and $\kappa_\ell < \frac{\tau - \bar{\tau}}{\bar{d} - d} < \kappa_h$, where $\tau - \bar{\tau}$ and $\bar{d} - d$ can be made arbitrarily close to 0. By the same reasoning as in

[30] To show that O_2 is not empty, one can construct an element $(q, d, \tau) \in \mathcal{F}$ such that $\tau = \bar{\tau}$ and $\kappa_\ell < \frac{u(\bar{q}) - u(q)}{\bar{d} - d} < \kappa_h$ where $u(\bar{q}) - u(q) > 0$ and $\bar{d} - d > 0$ can be made arbitrarily small. To show that $(\bar{q}, \bar{d}, \bar{\tau})$ is in the closure of O_2, consider a sequence $\{(q_n, d_n, \tau_n)\}_{n=1}^\infty$ such that $\tau_n = \bar{\tau}$, $u(\bar{q}) - u(q_n) > 0$, $\bar{d} - d_n > 0$ and $\kappa_\ell < \frac{u(\bar{q}) - u(q_n)}{\bar{d} - d_n} < \kappa_h$ for all $n \in \mathbb{N}$ and $(q_n, d_n) \to (\bar{q}, \bar{d})$. All the terms of the sequence are in O_2 and it converges to $(\bar{q}, \bar{d}, \bar{\tau})$.

(i), there is $(\tilde{y}, \tilde{d}, \tilde{\tau}) \in O_1 \cap O_2$ that satisfies (14)-(16) with $\chi = \ell$, and the proposed pooling equilibrium fails the Intuitive Criterion. ■

Proof of Lemma 2. **(i) Offer by the ℓ-type buyer.** Provided that $d \geq 0$, any offer that satisfies (18) is acceptable since

$$-c(y) + \{\lambda(y,d,\tau)\kappa_h + [1-\lambda(y,d,\tau)]\,\kappa_\ell\}\,d + \tau \geq -c(y) + \kappa_\ell d + \tau \geq 0,$$

for all $\lambda(y,d,\tau) \in [0,1]$. Moreover, the complete information offer is such that $\kappa_\ell d + \tau \geq 0$, and hence the requirement $d \geq 0$ is not binding. Consequently, the complete information payoff for the ℓ-type buyer can always be achieved. A payoff strictly greater than the complete information payoff is obtained only if (18) is violated, i.e.,

$$-c(y) + \kappa_\ell d + \tau < 0. \tag{53}$$

But an offer is acceptable if

$$-c(y) + \{\lambda(y,d,\tau)\kappa_h + [1-\lambda(y,d,\tau)]\,\kappa_\ell\}\,d + \tau \geq 0. \tag{54}$$

From (53) and (54), $\kappa_\ell d < \{\lambda(y,d,\tau)\kappa_h + [1-\lambda(y,d,\tau)]\,\kappa_\ell\}\,d$, and hence $d > 0$ and $\lambda(y,d,\tau) > 0$. This has been ruled out by Lemma 1.

Finally, I rule out offers such that $d_\ell < 0$. The complete information offer is consistent with $d_\ell < 0$ if and only if $z^b + \kappa_\ell a^b > c(y^*)$, in which case $y_\ell = y^*$. The payoff of an h-type buyer who offers $(y_\ell, d_\ell, \tau_\ell)$ is

$$u(y_\ell) - \kappa_h d_\ell - \tau_\ell = u(y^*) - c(y^*) - (\kappa_h - \kappa_\ell)d_\ell > u(y^*) - c(y^*),$$

if $d_\ell < 0$. So the h-type buyer could obtain more than the total surplus of the match, which would violate the incentive-compatibility condition in any separating equilibrium.

(ii) Offer by the h-type buyer. Suppose there is an equilibrium where the ℓ-type buyer achieves his complete information payoff, $U_\ell^b = u(y_\ell) - c(y_\ell)$, and the expected payoff of the h-type is $U_h^b \in [0, \bar{U})$, where $\bar{U}$ is the payoff associated with the solution to (20)-(23). For $\varepsilon > 0$, define U^ε as

$$U^\varepsilon = \max_{y,\tau,d}\,[u(y) - \kappa_h d - \tau] \tag{55}$$

$$\text{s.t.} \quad -c(y) + \kappa_h d + \tau \geq 0 \tag{56}$$

$$u(y) - \kappa_\ell d - \tau \leq U_\ell^b - \varepsilon \tag{57}$$

$$-z^s \leq \tau \leq z^b, \quad -a^s \leq d \leq a^b. \tag{58}$$

The set of acceptable and feasible offers is compact, and it is nonempty provided that $\varepsilon < U_\ell^b$. From the Theorem of the Maximum, U^ε is continuous in ε, and $\lim_{\varepsilon \to 0} U^\varepsilon = \bar{U}$. Hence, there is an $\varepsilon > 0$ such that $U^\varepsilon > U_h^b$. The associated offer satisfies (14)-(16), i.e., the proposed equilibrium violates the Intuitive Criterion.

Suppose that $U_h^b > \bar{U}$. Then, either the seller's participation constraint, (21), or the incentive-compatibility constraint, (22), are violated. Suppose $-c(y) + \kappa_h d + \tau < 0$. The offer is acceptable if (54) holds, which implies $\kappa_h d < \{\lambda(y,d,\tau)\kappa_h + [1 - \lambda(y,d,\tau)]\,\kappa_\ell\}\, d$, and hence $d < 0$ and $\lambda(y,d,\tau) < 1$. This has been ruled out by Lemma 1. If $u(y) - \kappa_\ell d - \tau > U_\ell^b$, then ℓ-type buyers can achieve a payoff strictly greater than their complete information payoff, which contradicts (i).

Finally, the solution to (20)-(23) is incentive-compatible since $(y_\ell, d_\ell, \tau_\ell)$ with $d_\ell \geq 0$ satisfies (21) and (22).

(iii) Beliefs.

A belief system consistent with the offers in Lemma 2 is such that $\lambda(y_h, d_h, \tau_h) = 1$ and $\lambda(y_\ell, d_\ell, \tau_\ell) = 0$ if $(y_h, d_h, \tau_h) \neq (y_\ell, d_\ell, \tau_\ell)$, and $\lambda(y_h, d_h, \tau_h) = \pi_h$ and $\lambda(y_\ell, d_\ell, \tau_\ell) = \pi_\ell$ if $(y_h, \tau_h) = (y_\ell, \tau_\ell)$ and $d_h = d_\ell = 0$ (from Bayes' rule). It will be established in Proposition 1 that the offers (y_h, d_h, τ_h) and $(y_\ell, d_\ell, \tau_\ell)$ are distinct unless $d_h = d_\ell = 0$. For out-of-equilibrium offers,

$$\begin{aligned} \lambda(y,d,\tau) &= 1 \text{ if (22) holds} \\ &= 0 \text{ otherwise.} \end{aligned}$$

Offers that violate (22) also violate (18), and since they are attributed to ℓ-type buyers, they are rejected.
■

Proof of Proposition 1. The buyer's objective function in (20) is continuous, and it is maximized over a non-empty, compact set. Hence, by the Theorem of the Maximum, there is a solution to (20)-(23). If $a^b = 0$, then the maximization to (20) subject to (21) gives $y_h = \min\left[y^*, c^{-1}(z^b)\right] = y_\ell$, $\tau_h + \kappa_h d_h = c(y_h)$, and $d_h \leq 0$. The incentive-compatibility condition (22) implies

$$u(y_h) - c(y_h) + (\kappa_h - \kappa_\ell) d_h \leq u(y_\ell) - c(y_\ell),$$

which is satisfied. This solution is consistent with Parts 1 and 2 of the Proposition. In the following, I focus on the case where $a^b > 0$.

Part 1 of the Proposition. I investigate in turn the conditions under which the constraints (21) and (22) are slack. First, suppose that the incentive-compatibility condition (22) is slack. Then, $y_h = \min\left[y^*, c^{-1}(\kappa_h a^b + z^b)\right] \geq y_\ell$. Since from (21) $c(y_h) = \kappa_h d_h + \tau_h$, then (22) becomes

$$u(y_h) - c(y_h) + d_h(\kappa_h - \kappa_\ell) \leq u(y_\ell) - c(y_\ell).$$

If $z^b < c(y^*)$, then $d_h > 0$ and (22) is violated, which is a contradiction. If $z^b \geq c(y^*)$, then $y_h = y_\ell = y^*$ and the inequality above implies $d_h \leq 0$, as in Part 1 of the Proposition.

Second, suppose that the seller's participation constraint (21) is slack. Substitute $u(y_h)$ by its expression given by (22) at equality into the objective function (20) to get

$$U_h^b = \max_{d \in [-a^s, a^b]} \left[(\kappa_\ell - \kappa_h)\, d + U_\ell^b\right] = U_\ell^b + (\kappa_h - \kappa_\ell) a^s,$$

and $d_h = -a^s$. If $a^s > 0$, then $u(y_h) - c(y_h) \geq U_h^b > U_\ell^b = u(y_\ell) - c(y_\ell)$, which requires $y_\ell < y^*$ and $y_h > y_\ell$. From (18) and (21), $\kappa_h d_h + \tau_h \geq c(y_h) > c(y_\ell) = z^b + \kappa_\ell a^b$, and hence $\tau_h > z^b + \kappa_\ell a^b + \kappa_h a^s$. This inequality violates feasibility. If $a^s = 0$, then $U_h^b = U_\ell^b = u(y_\ell) - c(y_\ell)$. Since $d_h = 0$, $U_h^b \leq \max[u(y) - \tau]$ subject to $-c(y) + \tau = 0$. Hence, $U_h^b = U_\ell^b$ if and only if $z^b \geq c(y^*)$. In that case, $y_h = y^*$ and $\tau_h = c(y^*)$, which is consistent with Part 1 of the Proposition.

Part 2 of the Proposition. I show first that the constraint $\tau_h \leq z^b$ is binding when $z^b < c(y^*)$. If $\tau_h \leq z^b$ is slack, then $y_h = y^*$, $d_h \leq 0$ and $\tau_h + \kappa_h d_h = c(y^*)$. This solution maximizes (20) subject to (21), and the constraint $d_h \leq 0$ guarantees that (22) holds. However, $\tau_h \geq c(y^*)$ is in contradiction with $z^b < c(y^*)$.

Since (21) is binding and $\tau_h = z^b$, d_h is given by (27). Substitute d_h by its expression into (22) at equality to get (28). For all $y_h \in [0, y_\ell]$ the left side of (28) is strictly increasing. It is nonpositive at $y_h = 0$, and greater than $u(y_\ell) - c(y_\ell)$ at $y_h = y_\ell$ if $c(y_\ell) > z^b$. This last inequality holds from (17)-(19). Indeed, if $z^b < c(y^*)$, then $c(y_\ell) = \min\left[c(y^*), z^b + \kappa_\ell a^b\right] > z^b$ since I focus on the case $a^b > 0$. Hence, there is a unique $y_h \in (0, y_\ell)$ solution to (28). The objective in (20) $u(y_h) - c(y_h) = u(y_\ell) - c(y_\ell) - \left(1 - \frac{\kappa_\ell}{\kappa_h}\right)\left[c(y_h) - z^b\right]$ is decreasing in y_h for any solution to (28). Hence, the unique solution in $(0, y_\ell)$ delivers a maximum to the problem (20)-(23). Given a unique y_h, d_h is determined by (27). Finally, $c(y_h) = z^b + \kappa_h d_h < c(y_\ell) = \tau_\ell + \kappa_\ell d_\ell \leq z^b + \kappa_\ell a^b$ implies $d_h < a^b$. From (28), $y_h < y_\ell$ implies $c(y_h) - z^b > 0$ and, from (27), $d_h > 0$. ■

Proof of Proposition 2.

(i) From Proposition 1, if $z^b < c(y^*)$, then y_h is the unique solution in $[0, y_\ell]$ to (28). Differentiate (28) to obtain

$$\frac{\partial y_h}{\partial \kappa_h} = -\frac{\frac{\kappa_\ell}{\kappa_h} d_h}{u'(y_h) - \frac{\kappa_\ell}{\kappa_h} c'(y_h)} < 0,$$
$$\frac{\partial y_h}{\partial \kappa_\ell} = \frac{\left[u'(y_\ell)/c'(y_\ell) - 1\right] a^b + d_h}{u'(y_h) - \frac{\kappa_\ell}{\kappa_h} c'(y_h)} > 0,$$

where $y_\ell = \min\left[y^*, c^{-1}(\kappa_\ell a^b + z^b)\right]$ and $d_h > 0$ (from Proposition 1 and the assumption $a^b > 0$). From (27), $\frac{\partial d_h}{\partial \kappa_\ell} = \frac{c'(y_h)}{\kappa_h} \frac{\partial y_h}{\partial \kappa_\ell} > 0$ and

$$\frac{\partial d_h}{\partial \kappa_h} = -\frac{u'(y_h) d_h}{\kappa_h u'(y_h) - \kappa_\ell c'(y_h)} < 0.$$

Since $\pi_h \kappa_h + \pi_\ell \kappa_\ell = 1$, then

$$\frac{d(d_h)}{d\kappa_h} = \frac{\partial d_h}{\partial \kappa_h} - \frac{\pi_h}{\pi_\ell} \frac{\partial d_h}{\partial \kappa_\ell} < 0.$$

(ii) From (28), as κ_ℓ approaches to 0 y_h tends to the solution to

$$u(y_h) - z^b = u(y_\ell) - c(y_\ell) = u(y_\ell) - z^b,$$

where I have used that $y_\ell = c^{-1}(z^b)$ when $z^b < c(y^*)$. Consequently, $y_h \to y_\ell$ and, from (27),

$$d_h = \frac{c(y_h) - z^b}{\kappa_h} \to 0.$$

■

Proof of Proposition 3. From Proposition 1, if $z^b < c(y^*)$, then y_h is the unique solution in $[0, y_\ell]$ to (28). Differentiating (28),

$$\frac{\partial y_h}{\partial z^b} = \frac{u'(y_\ell)/c'(y_\ell) - \frac{\kappa_\ell}{\kappa_h}}{u'(y_h) - \frac{\kappa_\ell}{\kappa_h} c'(y_h)} > 0.$$

From (27), $\frac{\partial(\kappa_h d_h)}{\partial z^b} = c'(y_h) \frac{\partial y_h}{\partial z^b} - 1$, and hence (29). The assumption $a^b > 0$ implies $y_\ell > y_h$ (Proposition 1) and $\frac{\partial(\kappa_h d_h)}{\partial z^b} < 0$. The expression for $\frac{\partial d_h}{\partial a^b}$ in (30) is obtained by a similar reasoning. ■

Proof of Lemma 4. I show that the objective function in (33) is jointly concave in (z, a). First, compute the first and second partial derivatives and the cross-partial derivatives of the surplus functions $S^\ell(z, a)$ and $S^h(z, a)$. From Lemma 2, $S^\ell(z, a) = \hat{\mathcal{S}}_\ell(z + \kappa_\ell a)$ with

$$\hat{\mathcal{S}}_\ell(z + \kappa_\ell a) = u \circ c^{-1}(z + \kappa_\ell a) - z - \kappa_\ell a \quad \text{if } z + \kappa_\ell a < c(y^*)$$
$$= u(y^*) - c(y^*) \quad \text{otherwise.}$$

Therefore, $S_a^\ell = \kappa_\ell \hat{\mathcal{S}}_\ell'$, $S_z^\ell = \hat{\mathcal{S}}_\ell'$, $S_{zz}^\ell = \hat{\mathcal{S}}_\ell''$, $S_{az}^\ell = \kappa_\ell \hat{\mathcal{S}}_\ell''$, and $S_{aa}^\ell = (\kappa_\ell)^2 \hat{\mathcal{S}}_\ell''$. From Proposition 1, if $z < c(y^*)$, then y_h solves (28), i.e.,

$$u(y_h) - c(y_h) + \left(1 - \frac{\kappa_\ell}{\kappa_h}\right)[c(y_h) - z] = \hat{\mathcal{S}}_\ell(z + \kappa_\ell a).$$

Totally differentiating the equation above,

$$\begin{aligned} \left[u'(y_h) - \frac{\kappa_\ell}{\kappa_h} c'(y_h)\right] \frac{\mathtt{d} y_h}{\mathtt{d} z} &= 1 - \frac{\kappa_\ell}{\kappa_h} + \hat{\mathcal{S}}_\ell' \\ \left[u'(y_h) - \frac{\kappa_\ell}{\kappa_h} c'(y_h)\right] \frac{\mathtt{d} y_h}{\mathtt{d} a} &= \kappa_\ell \hat{\mathcal{S}}_\ell'. \end{aligned}$$

Notice that $\frac{\mathtt{d} y_h}{\mathtt{d} z} > 0$ for all $z < c(y^*)$, and $\frac{\mathtt{d} y_h}{\mathtt{d} a} > 0$ for all (z, a) such that $z + \kappa_\ell a < c(y^*)$. From Proposition 1, the seller's participation constraint (21) holds at equality so that $S^h(z, a) = u(y_h) - c(y_h)$. Hence,

$$\begin{aligned} S_z^h(z, a) &= [u'(y_h) - c'(y_h)] \frac{\mathtt{d} y_h}{\mathtt{d} z} = \Delta(y_h)\left(1 - \frac{\kappa_\ell}{\kappa_h} + \hat{\mathcal{S}}_\ell'\right) \\ S_a^h(z, a) &= [u'(y_h) - c'(y_h)] \frac{\mathtt{d} y_h}{\mathtt{d} a} = \Delta(y_h) \kappa_\ell \hat{\mathcal{S}}_\ell' \end{aligned}$$

where

$$\Delta(y) \equiv \frac{u'(y) - c'(y)}{u'(y) - \frac{\kappa_\ell}{\kappa_h} c'(y)} = 1 - \frac{1 - \frac{\kappa_\ell}{\kappa_h}}{u'(y)/c'(y) - \frac{\kappa_\ell}{\kappa_h}}.$$

For all $y \in [0, y^*]$, $\Delta(y) \in [0, 1]$ and, since $u'(y)/c'(y)$ is decreasing in y, $\Delta'(y) < 0$. Furthermore,

$$\begin{aligned} S_{zz}^h &= \Delta'(y_h) \frac{\mathtt{d} y_h}{\mathtt{d} z}\left(1 - \frac{\kappa_\ell}{\kappa_h} + \hat{\mathcal{S}}_\ell'\right) + \Delta(y_h) \hat{\mathcal{S}}_\ell'' \\ S_{za}^h &= \Delta'(y_h) \frac{\mathtt{d} y_h}{\mathtt{d} z} \kappa_\ell \hat{\mathcal{S}}_\ell' + \Delta(y_h) \kappa_\ell \hat{\mathcal{S}}_\ell'' \\ &= \Delta'(y_h) \frac{\mathtt{d} y_h}{\mathtt{d} a}\left(1 - \frac{\kappa_\ell}{\kappa_h} + \hat{\mathcal{S}}_\ell'\right) + \Delta(y_h) \kappa_\ell \hat{\mathcal{S}}_\ell'' \\ S_{aa}^h &= \Delta'(y_h) \frac{\mathtt{d} y_h}{\mathtt{d} a} \kappa_\ell \hat{\mathcal{S}}_\ell' + \Delta(y_h)(\kappa_\ell)^2 \hat{\mathcal{S}}_\ell''. \end{aligned}$$

For all $z < c(y^*)$, $S_{zz}^h < 0$. Consequently, the first leading principal minor of the Hessian matrix associated with (33), $\pi_h S_{zz}^h + \pi_\ell S_{zz}^\ell$, is nonpositive, and it is strictly negative for all $z < c(y^*)$.

The determinant of the Hessian matrix associated with (33) is

$$|\mathbb{H}| = \left(\pi_h S_{zz}^h + \pi_\ell S_{zz}^\ell\right)\left(\pi_h S_{aa}^h + \pi_\ell S_{aa}^\ell\right) - \left(\pi_h S_{za}^h + \pi_\ell S_{za}^\ell\right)^2.$$

It can be decomposed as $|\mathbb{H}| = \Gamma_1 + \Gamma_2 + \Gamma_3$ where

$$\begin{aligned}\Gamma_1 &= (\pi_\ell)^2 \left[S^\ell_{zz} S^\ell_{aa} - \left(S^\ell_{za}\right)^2\right] \\ \Gamma_2 &= (\pi_h)^2 \left[S^h_{zz} S^h_{aa} - \left(S^h_{za}\right)^2\right] \\ \Gamma_3 &= \pi_h \pi_\ell \left[S^h_{zz} S^\ell_{aa} + S^\ell_{zz} S^h_{aa} - 2 S^h_{za} S^\ell_{za}\right].\end{aligned}$$

Since $S^\ell(z,a) = \hat{\mathcal{S}}_\ell(z + \kappa_\ell a)$, $\Gamma_1 = 0$. After some calculation,

$$\begin{aligned}\Gamma_2 &= (\pi_h)^2 \left(1 - \frac{\kappa_\ell}{\kappa_h}\right) \Delta \Delta' \hat{\mathcal{S}}''_\ell \kappa_\ell \left(\frac{\mathtt{d} y_h}{\mathtt{d} z}\kappa_\ell - \frac{\mathtt{d} y_h}{\mathtt{d} a}\right) \\ \Gamma_3 &= \pi_h \pi_\ell \left(1 - \frac{\kappa_\ell}{\kappa_h}\right) \Delta' \hat{\mathcal{S}}''_\ell \kappa_\ell \left(\frac{\mathtt{d} y_h}{\mathtt{d} z}\kappa_\ell - \frac{\mathtt{d} y_h}{\mathtt{d} a}\right),\end{aligned}$$

where Δ and Δ' are evaluated at $y = y_h$. Therefore,

$$|\mathbb{H}| = \left(\frac{\mathtt{d} y_h}{\mathtt{d} z}\kappa_\ell - \frac{\mathtt{d} y_h}{\mathtt{d} a}\right)\left(1 - \frac{\kappa_\ell}{\kappa_h}\right) \Delta' \hat{\mathcal{S}}''_\ell \kappa_\ell \pi_h \left(\pi_h \Delta + \pi_\ell\right),$$

with

$$\kappa_\ell \frac{\mathtt{d} y_h}{\mathtt{d} z} - \frac{\mathtt{d} y_h}{\mathtt{d} a} = \left[u'(y_h) - \frac{\kappa_\ell}{\kappa_h} c'(y_h)\right]^{-1} \kappa_\ell \left(1 - \frac{\kappa_\ell}{\kappa_h}\right) > 0, \quad \forall y_h \leq y^*.$$

Hence, $|\mathbb{H}| \geq 0$, with a strict inequality for all $z + \kappa_\ell a < c(y^*)$. Hence, the first-order conditions (38) and (39) are necessary and sufficient for an optimum to the buyer's problem.

In the following, I review the different cases depending on whether asset prices are equal, greater, or smaller than their fundamental values (β).

(i) $q_z > \beta$ and $q_a > \beta$.

I now show that there is a unique solution to (33). First, the solution to (33) is such that $z + \kappa_\ell a \leq c(y^*)$. Suppose $z + \kappa_\ell a > c(y^*)$. Then, $\hat{\mathcal{S}}'_\ell = 0$ and $S^h_a(z,a) = S^\ell_a(z,a) = 0$. The first-order condition for a, (39), implies $a = 0$. If $z > c(y^*)$, then $y_h = y_\ell = y^*$ and hence $S^h_z(z,a) = S^\ell_z(z,a) = 0$. The first-order condition for z, (38), implies $z = 0$. A contradiction.

So one can restrict (z,a) to the compact set $\{(z,a) \in \mathbb{R}_{2+} : z + \kappa_\ell a \leq c(y^*)\}$ and, from the Theorem of the Maximum, a solution to (33) exists, and it satisfies the first-order conditions (38)-(39). Since $\mathbb{H}$ is negative definite for all (z,a) such that $z + \kappa_\ell a < c(y^*)$, i.e., the leading principal minors of $\mathbb{H}$ alternate in sign with the first one being negative, the solution to (33) is unique.

(ii) $q_z > \beta$ and $q_a = \beta$.

From the first-order condition for a, (39), $S_a^h(z,a) = S_a^\ell(z,a) = 0$, which requires $z + \kappa_\ell a \geq c(y^*)$. The first-order condition for z, (38), implies

$$-\left(\frac{q_z - \beta}{\beta}\right) + \pi_h \Delta\left[y_h(z)\right]\left(1 - \frac{\kappa_\ell}{\kappa_h}\right) \leq 0, \quad \text{“} = \text{”} \quad \text{if } z > 0, \tag{59}$$

where I have used that $\hat{\mathcal{S}}_\ell' = 0$. From (28), $y_h(z)$ is implicitly defined by

$$u(y_h) - c(y_h) + \left(1 - \frac{\kappa_\ell}{\kappa_h}\right)\left[c(y_h) - z\right] = u(y^*) - c(y^*) \ \text{ if } \ z < c(y^*), \tag{60}$$

and $y_h(z) = y^*$ if $z \geq c(y^*)$. For all $z \geq c(y^*)$, $\Delta(y_h) = \Delta(y^*) = 0$, and (59) implies $z = 0$. A contradiction. Since $\Delta' < 0$ and $y_h'(z) > 0$ for all $z \in (0, c(y^*))$, and since the function on the left side of (59) is continuous in z, there is a unique $z \in [0, c(y^*)]$ solution to (59). Consequently, $a \in [\frac{c(y^*)-z}{\kappa_\ell}, \infty)$.

(iii) $q_z = \beta$ and $q_a > \beta$.

From (38) $S_z^h(z,a) = S_z^\ell(z,a) = 0$, which implies $z \geq c(y^*)$. Then, $S_a^h(z,a) = S_a^\ell(z,a) = 0$ and, from (39), $a = 0$.

(iv) $q_z = q_a = \beta$.

From (38), $S_z^h(z,a) = S_z^\ell(z,a) = 0$, which implies $z \geq c(y^*)$. From (39), $a \in [0, +\infty)$.

(v) $q_z < \beta$ or $q_a < \beta$**.**

Since $S_a^h(z,a) \geq 0$ and $S_a^\ell(z,a) \geq 0$ there is no solution to the first-order condition for a, (39). ■

Proof of Proposition 4. The proof proceeds in two parts. First, I show that the asset prices are uniquely determined. Then, I prove that the output levels in the DM are also uniquely characterized.

(i) Asset prices

Consider the portfolio correspondance $P(q_z, q_a)$ defined as

$$P \equiv \left\{ \left(\int_{j \in \mathcal{J}} a(j)dj, \int_{j \in \mathcal{J}} z(j)dj \right) : [a(j), z(j)] \text{ solution to (33) if } j \in \mathcal{B} \text{ and to (35) if } j \in \mathcal{S} \right\}$$

The market-clearing conditions (36) and (37) require $(A, Z) \in P(q_z, q_a)$. From Lemma 4, for all (q_z, q_a) such that $q_z > \beta$ and $q_a > \beta$, $P(q_z, q_a) = (a^b, z^b)$ where (a^b, z^b) is the unique solution to (38)-(39). If $q_z > \beta$ and $q_a = \beta$, then $P(q_z, q_a) = \{z^b\} \times \left[\frac{c(y^*)-z^b}{\kappa_\ell}, \infty\right)$ where z^b is the unique solution to (38). If $q_z = \beta$ and $q_a = \beta$, then $P(q_z, q_a) = [c(y^*), \infty) \times [0, \infty)$. In all cases, the aggregate portfolio correspondance P coincides with the buyer's portfolio correspondance. Since the buyer's objective function in (33) is jointly concave in (z, a), then P is convex-valued. Consequently, $(A,Z) \in P(q_z,q_a)$ if and only if (A,Z) is solution to (33), i.e., (38) and (39) hold at equality with $a = A$ and $z = Z$.

From (38),

$$q_z = \beta \left[1 + \pi_h S_z^h (Z,A) + \pi_\ell S_z^\ell (Z,A)\right]. \tag{61}$$

Consequently, there is a unique q_z that solves (61), and it is such that $\frac{q_z}{\beta} \geq 1$. This expression for q_z coincides with (43) where the term in brackets is $\mathcal{L}_z$ given by (45). From (39),

$$q_a = \beta \left[1 + \pi_h S_a^h (Z,A) + \pi_\ell S_a^\ell (Z,A)\right]. \tag{62}$$

Hence, q_a is uniquely determined, and it is such that $\frac{q_a}{\beta} \geq 1$. This expression for q_a coincides with (44) where the term in brackets is $\mathcal{L}_a$ given by (46).

(ii) DM allocations.

From (i) q_a and q_z are unique. From Lemma 4, if $q_a > \beta$ and $q_z > \beta$, then there is a unique solution to (33). From Proposition 1, if $\kappa = \kappa_h$, then (y_h, d_h, τ_h) is unique; if $\kappa = \kappa_\ell$, then y_ℓ and $\tau_\ell + \kappa_\ell d_\ell$ are uniquely determined. If $q_a = \beta$ and $q_z > \beta$, then $a(j)$ can vary across buyers and sellers but z^b is unique, and $z^b + \kappa_\ell a(j) \geq c(y^*)$ for all $j \in \mathcal{B}$ (see Lemma 4). Consequently, $y_\ell = y^*$ and, from (28), y_h is independent of $a(j)$ for all $j \in \mathcal{B}$, and it solves

$$u(y_h) - c(y_h) + \left(1 - \frac{\kappa_\ell}{\kappa_h}\right)\left[c(y_h) - z^b\right] = u(y^*) - c(y^*).$$

(60). If $q_a = q_z = \beta$, then $z^b \geq c(y^*)$ for all $j \in \mathcal{B}$ and hence $y_h = y_\ell = c(y^*)$. ■

Proof of Proposition 5. From the proof of Proposition 4, q_z and q_a solve (38) and (39) at equality with $a = A$ and $z = Z$.

(i) From (45), $\mathcal{L}_z = 0$ and $q_z = \beta$ if and only if $y_h = y_\ell = y^*$. From Proposition 1, $y_h = y^*$ requires $z^b = Z \geq c(y^*)$.

(ii) From (46), $\mathcal{L}_a = 0$ and $q_a = \beta$ if and only if $y_\ell = y^*$. This requires $\kappa_\ell a^b + z^b \geq c(y^*)$ or, from market-clearing, $\kappa_\ell A + Z \geq c(y^*)$.

(iii) From (45) and (46),

$$\begin{aligned}
\mathcal{L}_z &= \pi_h \Delta(y_h)\left(1 - \frac{\kappa_\ell}{\kappa_h} + \hat{\mathcal{S}}_\ell'\right) + \pi_\ell \hat{\mathcal{S}}_\ell' \\
\mathcal{L}_a &= \kappa_\ell \left[\pi_h \Delta(y_h)\hat{\mathcal{S}}_\ell' + \pi_\ell \hat{\mathcal{S}}_\ell'\right].
\end{aligned}$$

Since $\kappa_\ell < 1$, $\mathcal{L}_a \leq \mathcal{L}_z$. Moreover, the inequality is strict unless $\mathcal{L}_a = \mathcal{L}_z = 0$. Hence, from (i) and (ii), $\mathcal{L}_a < \mathcal{L}_z$ if and only if $Z < Z^*$. From (38)-(39),

$$\begin{aligned} r_z &= \frac{1}{\beta(1+\mathcal{L}_z)} \\ r_a &= \frac{1}{\beta(1+\mathcal{L}_a)}. \end{aligned}$$

Hence, $r_a \geq r_z$. Moreover, the inequality is strict unless $r_a = r_z = \beta^{-1}$. Hence, from (i) and (ii), $r_a > r_z$ if and only if $Z < Z^*$. ■

Proof of Proposition 6. I first establish that $q_z > q_z^{ci}$ if $Z < Z^*$. From Proposition 1, if $Z < Z^*$, then $y_h < y_\ell = \min\left[y^*, c^{-1}(Z + \kappa_\ell A)\right]$ in the private-information economy. In the complete-information economy, $y_h^{ci} = \min\left[y^*, c^{-1}(Z + \kappa_h A)\right] \geq y_\ell^{ci} = \min\left[y^*, c^{-1}(Z + \kappa_\ell A)\right]$. Hence, $y_h < y_\ell = y_\ell^{ci} \leq y_h^{ci}$. From (43) and (45),

$$\frac{q_z}{\beta} = 1 + \pi_\ell\left[\frac{u'(y_\ell)}{c'(y_\ell)} - 1\right] + \pi_h\left[\frac{\kappa_h u'(y_\ell)/c'(y_\ell) - \kappa_\ell}{\kappa_h + \frac{\kappa_h - \kappa_\ell}{u'(y_h)/c'(y_h) - 1}}\right],$$

which is decreasing with y_h. Consequently,

$$\frac{q_z}{\beta} > \frac{u'(y_\ell)}{c'(y_\ell)} \geq \frac{q_z^{ci}}{\beta}.$$

Next, I turn to the second part of the Proposition. From (47) and (48),

$$\frac{q_z^{ci} - q_a^{ci}}{\beta} = \pi_\ell(1-\kappa_\ell)\left[\frac{u'(y_\ell^{ci})}{c'(y_\ell^{ci})} - 1\right] - \pi_h(\kappa_h - 1)\left[\frac{u'(y_h^{ci})}{c'(y_h^{ci})} - 1\right].$$

From (43) and (44),

$$\frac{q_z - q_a}{\beta} = \mathcal{L}_z - \mathcal{L}_a$$

where

$$\begin{aligned} \mathcal{L}_z - \mathcal{L}_a &= \pi_\ell(1-\kappa_\ell)\left[\frac{u'(y_\ell)}{c'(y_\ell)} - 1\right] + \pi_h\left[\frac{\frac{u'(y_h)}{c'(y_h)} - 1}{\kappa_h\frac{u'(y_h)}{c'(y_h)} - \kappa_\ell}\right] \times \\ &\quad \left[(1-\kappa_\ell)\kappa_h\frac{u'(y_\ell)}{c'(y_\ell)} + (\kappa_h - 1)\kappa_\ell\right] \end{aligned}$$

Since $y_\ell = y_\ell^{ci}$ and

$$-\pi_h(\kappa_h - 1)\left[\frac{u'(y_h^{ci})}{c'(y_h^{ci})} - 1\right] \leq 0$$

$$\pi_h\left[\frac{\frac{u'(y_h)}{c'(y_h)} - 1}{\kappa_h\frac{u'(y_h)}{c'(y_h)} - \kappa_\ell}\right]\left[(1-\kappa_\ell)\kappa_h\frac{u'(y_\ell)}{c'(y_\ell)} + (\kappa_h - 1)\kappa_\ell\right] > 0,$$

then $q_z - q_a > q_z^{ci} - q_a^{ci}$. ■

Proof of Proposition 7. (i) From (28), as $\kappa_\ell/\kappa_h \to 1$, then $y_h \to y_\ell$. From (43) and (44),

$$r_z = r_a = \frac{c'(y)}{\beta u'(y)},$$

where $y = \min\left[y^*, c^{-1}(A+Z)\right]$.

(ii) From the proof of Lemma 4, as $\kappa_\ell \to 0$, then $S_a^h(Z,A), S_a^\ell(Z,A) \to 0$. From (44), $q_a \to \beta$ and $r_a = \frac{1}{q_a} \to \beta^{-1}$. From Proposition 1, (43) and (45),

$$q_z \to \beta \frac{u'(y)}{c'(y)} \geq \beta,$$

with $y = \min\left[y^*, c^{-1}(Z)\right]$. ■

Proof of Proposition 8.

(i) Differentiating (61),

$$\frac{dq_z}{dZ} = \beta\left[\pi_h S_{zz}^h(Z,A) + \pi_\ell S_{zz}^\ell(Z,A)\right]. \tag{63}$$

From Proposition 5, $S_{zz}^h > 0$ if and only if $Z < Z^*$, in which case $\frac{dq_z}{dZ} < 0$, and hence $\frac{dr_z}{dZ} > 0$ and $\frac{d\mathcal{L}_z}{dZ} < 0$.

(ii) Differentiate (62) to obtain

$$\frac{dq_a}{dZ} = \beta\left[\pi_h S_{az}^h(Z,A) + \pi_\ell S_{az}^\ell(Z,A)\right].$$

If $Z + \kappa_\ell A < c(y^*)$, then S_{az}^h,$S_{az}^\ell < 0$ (from the proof of Lemma 4), and $\frac{dq_a}{dZ} < 0$. This implies $d\mathcal{L}_a/dZ < 0$ and $dr_a/dZ > 0$.

(iii) From (45) and (46),

$$\begin{aligned}\mathcal{L}_z - \mathcal{L}_a \quad = \quad & \pi_\ell(1-\kappa_\ell)\left[\frac{u'(y_\ell)}{c'(y_\ell)} - 1\right] + \pi_h\left[\frac{\frac{u'(y_h)}{c'(y_h)} - 1}{\kappa_h\frac{u'(y_h)}{c'(y_h)} - \kappa_\ell}\right] \times \\ & \left[(1-\kappa_\ell)\kappa_h\frac{u'(y_\ell)}{c'(y_\ell)} + (\kappa_h - 1)\kappa_\ell\right]\end{aligned}$$

The liquidity differential $\mathcal{L}_z - \mathcal{L}_a$ is strictly decreasing in y_h and y_ℓ. Since $\frac{dy_\ell}{dZ} > 0$ whenever $Z + \kappa_\ell A < c(y^*)$ and $\frac{dy_h}{dZ} > 0$ for all $Z < c(y^*)$ (from the proof of Lemma 4), then $\frac{\partial(\mathcal{L}_z - \mathcal{L}_a)}{\partial Z} < 0$ for all $Z < Z^*$. From (43) and (44), $1/r_z = \beta(1+\mathcal{L}_z)$ and $1/r_a = \beta(1+\mathcal{L}_a)$, which gives

$$\frac{1}{r_z} - \frac{1}{r_a} = \frac{r_a - r_z}{r_z r_a} = \beta\left(\mathcal{L}_z - \mathcal{L}_a\right)$$

(iv) From Proposition 1, $y_h = y^*$ and $y_\ell = \min[y^*, c^{-1}(Z + \kappa_\ell A)] = y^*$ if $Z \geq Z^* = c(y^*)$. From (43)-(46), it implies $q_z = q_a = \beta$. ■

Proof of Proposition 10. From (49),

$$i = \theta \left[\pi_h S_z^h (M + Z, A) + \pi_\ell S_z^\ell (M + Z, A)\right] + (1 - \theta) \left\{ \frac{u'\left[c^{-1}(M)\right]}{c'\left[c^{-1}(M)\right]} - 1 \right\}. \tag{64}$$

If $\theta < 1$, then the right side of (64) is decreasing in M: it tends to infinity as M approaches 0, and it is equal to 0 for all $M \geq c(y^*)$. Hence, for all $i > 0$, there is a $M \in (0, c(y^*))$ solution to the equation above. Given M, q_z and q_a are determined by (51) and (52).

From (45) and (51), $q_z > \beta$ (i.e., $\mathcal{L}_z > 0$) if and only if $M + Z < c(y^*)$ where M is the unique solution to (64). It can be checked from (64) that $M + Z$ is increasing with Z. Consequently, there is a threshold Z^{**} such that for all $Z < Z^{**}$, $q_z > \beta$. From (64), $Z^{**} = c(y^*) - \tilde{M}$ where $\tilde{M}$ solves $\frac{u'\left[c^{-1}(M)\right]}{c'\left[c^{-1}(M)\right]} = 1 + \frac{i}{1-\theta}$. If $Z < Z^{**}$, $\mathcal{L}_z > \mathcal{L}_a \geq 0$ and $r_z < r_a$. From (49) and (51),

$$\frac{\gamma}{\beta} = 1 + \theta \mathcal{L}_z + (1 - \theta)\mathcal{L}_m > 1 + \theta \mathcal{L}_z = \frac{q_z}{\beta},$$

since $\mathcal{L}_m > 0$. Consequently, $\gamma^{-1} < r_z$. ■

Proof of Proposition 11. From (64), if $Z < Z^{**}$, then $\frac{\partial M}{\partial Z} < 0$ and $\frac{\partial (M+Z)}{\partial Z} > 0$. If $Z > Z^{**}$, then $\frac{\partial M}{\partial Z} = 0$. From (51), if $Z < Z^{**}$, then $\frac{\partial q_z}{\partial Z} < 0$ and $\frac{\partial r_z}{\partial Z} < 0$ since $M + Z$ increases with Z, and hence $\mathcal{L}_z$ decreases with Z. ■