

SUB-OPTIMALITY OF THE FRIEDMAN RULE IN
TOWNSEND'S TURNPIKE AND LIMITED
COMMUNICATION MODELS OF MONEY: DO FINITE
LIVES AND INITIAL DATES MATTER?*

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Abstract

We construct an economy populated with infinitely-lived agents and show that the Friedman rule is suboptimal. We do that by showing that our economy and an overlapping generations model in which the Friedman rule is known to be suboptimal are homomorphic. We also discuss the importance of whether or not the economy has an initial date for this result.

Keywords: Friedman rule; monetary policy; overlapping generations; turnpike.

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1 Introduction

One of the big puzzles in monetary economics is the disparity between theory and practice concerning the optimum quantity of money. Theory has shown the Friedman rule to be optimal in many different environments and under many different assumptions (see, for example, Kimbrough 1986; Chari, Christiano, and Kehoe 1996; Correia and Teles 1996). Yet, in practice, no central bank (CB) states as its objective to implement the Friedman rule, and historical episodes in which deflation occurred and interest rates approached zero have often been considered very negative.¹

Overlapping generations models of money may help to bridge the gap between theory and practice. Researchers have shown that implementing the Friedman rule may not be the welfare-maximizing policy in some overlapping generations economies.² Recently, Paal and Smith (2000), Smith (2002 a and b) and Bhattacharya, Haslag, and Russell (2004) have offered explanations that account for why the Friedman rule is suboptimal in the overlapping generations (OG) framework. There is a common thread among these papers: the optimal policy for a CB is to equate the private cost of using money with the social cost of doing so. The point is that marginal social costs and marginal private costs are not equated at the Friedman rule in the economies studied by these authors.

Two important features differentiate the OG models in which the Friedman rule is suboptimal with the infinitely-lived representative-agent models in which the Friedman rule is optimal. First, in the OG models agents have finite lives. Second, in these models, money is used between agents who are, in some respect, heterogeneous. The heterogeneity comes from the fact that agents of different generations are alive in the same period and, because of it, may have different preferences about the optimal monetary policy in the current period. While it is clear that this kind of heterogeneity plays a crucial role for suboptimality of the Friedman rule, whether finite lives plays a role is as yet an unresolved

¹The Great Depression and Japan in the 1990s are two such episodes.

²See Helpman and Sadka (1979) and Wallace (1980), for instance.

issue.³ Whether finite lives matters is not merely of theoretical interest. If indeed if the length of the life does matter, we would expect most monetary models—in particular, ones in which agents are infinitely lived—to be inadequate to answer practical policy questions about the optimum quantity of money.⁴

In this paper, we construct an economy populated by infinitely-lived agents and show that the Friedman rule is suboptimal for exactly the same reasons as in the OG environment. To do this, we show that our economy and an OG economy in which the Friedman rule is suboptimal are homomorphic. We extend the analysis further to examine whether it matters if the economy has an initial date or not. We find that the treatment of an initial date is important, providing yet another set of conditions determining whether or not the Friedman rule is optimal. More importantly, this additional dimension permits a deeper look into the nature of monetary policy and the channels through which it affects welfare.

Our findings are important on two dimensions. First, we show that the length of live of agents is not important for the suboptimality of the Friedman rule. This finding, therefore, suggests there is nothing special about OG economies. Rather, the same frictions have the same effect in an economy with infinitely-lived and finitely-lived agents.

The second dimension involves the role played by an initial period in evaluating optimal monetary policies. In both the OG and the economy populated with infinitely-lived agents, markets are incomplete. But our results with and without the presence of an initial period suggests that market incompleteness is not a sufficient condition, just as finite lives is not sufficient, for the Friedman rule to be suboptimal. Rather, there is a mechanical interpretation: our results indicate the existence of an initial date is tantamount to a policymaker putting greater weight on the welfare of the initial moneyholders. The economic interpretation is straightforward: a change in the money growth rate has a first-order effect on

³Bhattacharya, Haslag, and Martin (2004) emphasize the role of heterogeneity. See also Haslag and Martin (2003).

⁴To be more precise, we are treating finite lives as models in which the agent's problem cannot be written as one in which the planning horizon is infinitely long. An OG economy with an operative bequest motive would be an example of a problem that is *de facto* an infinite horizon problem, for instance.

initial money holders because of the redistribution that occurs under the Friedman rule. If, however, there is no initial date, there is effectively no initial moneyholders and the Friedman rule is not the optimal policy.

The remainder of the paper proceeds as follows. Section 2 describes an OG environment in which the Friedman rule is suboptimal. Section 3 describes an economy with infinitely-lived agents and shows that it is homomorphic to the OG environment of section 2. Section 4 discusses the results and concludes.

2 A OG model with random relocation

Time is discrete. We consider two alternative economies: one with an initial date, in which time is denoted by $t = 1, 2, \dots$, and one with with no initial date, in which case time is denoted by $t = \dots, -1, 0, 1, \dots$. The world is divided into two spatially separated locations.⁵ Each location is populated by a continuum of agents of unit mass. Agents live for two periods, and receive an endowment of ω units of the single consumption good when young and nothing when old. Only old-age consumption is valued. Let c_t denote old-age consumption of the members of the generation born at date t ; their lifetime utility is given by $u(c_t) = \frac{c_t^{1-\rho}}{1-\rho}$, where $\rho \in (0, 1)$. If the economy has an initial date, then there also is an initial old generation whose members are endowed with an amount of cash M_0 .

After receiving their endowment and placing it into a bank, agents learn whether they must move to the other location or not. Let α denote the probability that an individual will be relocated. We assume a law of large number holds so α is also the measure of agents that are relocated. Further, we assume that the number of movers is symmetric; that is, α is the same on both islands. Upon learning their status, movers redeem their bank deposits in the form of money as this is the only way for them to acquire goods in the

⁵See Champ, Smith and Williamson (1996). We are using a simple modified version of their random-relocation model.

new location.⁶ In contrast, nonmovers redeem their deposits in the form of goods when old. Goods deposited in the bank can be used to acquire money from old agents belonging to the previous generation or put into storage. Each unit of the consumption good put into storage at date t yields $x > 1$ units of the consumption good at date $t + 1$, where x is a known constant.

The CB can levy lump-sum taxes τ on the endowment of agents by collecting the tax in the form of money balances removed from the economy. In contrast, a lump-sum subsidy is received in the form of a money injection. The money supply evolves according to $M_{t+1} = (1 + z) M_t$ and z is chosen by the CB in a manner that will be explained below. We assume $x \geq 1/(1 + z)$ implying that the return on money is no greater than the return on storage. Let p_t denote the time t price level; in steady states, $p_{t+1} = (1 + z) p_t$. Also, since we focus on steady-states, we drop the time subscript in what follows.

Agents deposit their entire after-tax/transfer endowments with a bank. The bank chooses the gross real return it pays to movers, d^m , and to nonmovers, d^n . In addition, the bank chooses values m (real value of money balances) and s (storage investment) respectively. These choices must satisfy the bank's balance sheet constraint

$$m + s \leq \omega - \tau. \tag{1}$$

Banks behave competitively, so they take as given the return on their investments. In particular, the return on real money balances is p_t/p_{t+1} . If $x > p_t/p_{t+1}$ banks will want to hold as little liquidity as possible since money is dominated in rate of return. If $x = p_t/p_{t+1}$, banks are indifferent between money and storage. In this case, we consider the limiting economy as $p_t/p_{t+1} \rightarrow x$.

Banks must have sufficient liquidity to meet the needs of movers. This is captured by the following expression:

$$\alpha d^m (\omega - \tau) \leq \frac{m}{1 + z}. \tag{2}$$

⁶The assumption that money is the only asset that can be traded between spatially and informationally separated locations is originally due to Townsend (1987)

A similar condition for non-movers, who consume all the proceeds from the storage technology, is given by

$$(1 - \alpha)d^m(\omega - \tau) \leq xs. \quad (3)$$

Banks maximize profits. Because of free entry, banks choose their portfolio, in equilibrium, in a way that maximizes the expected utility of a representative depositor. The bank's problem is written as

$$\max_{d^m, d^n} \frac{(\omega - \tau)^{1-\rho}}{1 - \rho} \left\{ \alpha (d^m)^{1-\rho} + (1 - \alpha) (d^n)^{1-\rho} \right\} \quad (4)$$

subject to equations (1), (2), and (3).

Let $\gamma \equiv m/(\omega - \tau)$ denote the bank's reserve-to-deposit ratio. Then, since equations (1), (2), and (3) hold with equality, the bank's objective function is to choose γ to maximize

$$\frac{(\omega - \tau)^{1-\rho}}{1 - \rho} \left\{ \alpha^\rho \left[\frac{\gamma}{1 + z} \right]^{1-\rho} + (1 - \alpha)^\rho [(1 - \gamma)x]^{1-\rho} \right\}. \quad (5)$$

Bhattacharya, Guzman, Huybens, and Smith (1997) show that the reserve to deposit ratio chosen by the bank is given by

$$\gamma(z) = \frac{1}{\left[1 + \frac{1-\alpha}{\alpha} \{(1 + z)x\}^{\frac{1-\rho}{\rho}} \right]} \quad (6)$$

and that it increases as $1 + z$ decreases.

2.1 The economy without an initial date

It is customary in this literature to ignore the initial old when considering the optimal monetary policy (see, for example, Schreft and Smith 2002). Hence, we first consider an economy without an initial date. Since, we consider steady state equilibria, the objective of the CB is to maximize the utility of a representative generation. Formally, the CB chooses the rate of growth of the money supply, z , to maximize (5), where γ is given by equation (6).

In this economy, it is optimal for the CB to maintain a constant money stock, as is stated in the following proposition.

Proposition 1 *The optimal rate of growth of the money supply in this economy is given by $z = 0$.*

Proof. See Appendix. ■

The Friedman rule is defined as the rate of growth of the money supply which equates the rate of return of money with the rate of return on storage. Formally, $z^{FR} = 1/x$. Proposition (1) states that the Friedman rule is not the optimal policy in this economy.

2.2 The economy with an initial date

We now consider an economy with an initial date. We interpret the existence of an initial date as a case in which there is a measure of agents identified as the initial old. For the initial old, consumption is equal to the real value of money balances. Let M_0 denote the quantity of nominal money balances held by a member of the initial old generation. Then, $c_1^0 = \frac{M_0}{p_1}$, where $p_1 = \frac{(1+z)M_0}{\gamma(\omega-\tau)}$. Note, in equilibrium, the reserve-to-deposit ratio and the lump-sum tax are functions of the money growth rate. At a steady state, the CB has the following objective function

$$W(z) = (1 - \beta) \frac{\left(\frac{M_0}{p_1}\right)^{1-\rho}}{1 - \rho} + \beta \frac{\{\Omega(z)\}^{1-\rho} \Gamma(z)}{1 - \rho} \quad (7)$$

where $\Omega(z) := \omega - \tau(z)$, and $\Gamma(z) := \alpha^\rho \left[\frac{\gamma(z)}{1+z}\right]^{1-\rho} + (1 - \alpha)^\rho [(1 - \gamma(z))x]^{1-\rho}$. The CB takes β as given and chooses z to maximize its objective function. We consider different values of the weight for the initial old generation and all other generations. For example, if $\beta = 0$, then the CB only considers the utility of the initial old. Conversely, as $\beta \rightarrow 1$, the weight of the initial old goes to zero and so the CB maximizes the utility of a representative generation (in steady states) and completely ignores the initial old.

Proposition 2 *The optimal rate of growth of the money supply in this economy is given by*

$$1 + z = 1 - \frac{1 - \beta}{1 - \beta(1 - \alpha^\rho)}, \quad (8)$$

subject to the constraint that $1 + z \geq \frac{1}{x}$.

Proof. See Appendix. ■

Given $\alpha > 0$, it is straightforward to see that if $\beta \rightarrow 1$, then $1 + z \rightarrow 1$. As $\beta \rightarrow 0$, in the limit the weight is all on the initial old; the constraint $1 + z \geq \frac{1}{x}$ eventually binds and the CB implements the Friedman rule.

Note that if $\beta x > 1$, the return on storage is so great, compared to the rate at which the CB weights the expected utility of future generations relative to the initial old, that the CB would like to accumulate assets so that future generations consume more than early generations. If we restrict our attention to values of β such that $\beta x \leq 1$, then the CB always chooses the Friedman rule. Indeed, we can rewrite equation (8) to get

$$1 + z = \frac{\alpha^\rho}{\frac{1}{\beta} - (1 - \alpha^\rho)}. \quad (9)$$

If $\beta x \leq 1$, then

$$\frac{\alpha^\rho}{\frac{1}{\beta} - (1 - \alpha^\rho)} \leq \frac{\alpha^\rho}{x - (1 - \alpha^\rho)} \leq \frac{1}{x}.$$

In many cases the CB would like to set a rate of growth of the money supply lower than the Friedman rule but this is not feasible since in that case nobody would store goods. So, for economies in which the CB does not prefer assets to accumulate over time, the best feasible policy is the Friedman rule.

3 A modified turnpike economy

In this section, we analyze a modified version of the model economy introduced by Townsend (1980). We show that our modified turnpike economy and the OG economy presented in the previous section are homomorphic. Accordingly, these economies share the property that the Friedman rule is *ex ante* suboptimal, even in a model in which agents are infinitely-lived.

As in the OG economies, time is discrete and we consider both a version in which there is an initial date and one without. The former is captured by $t = 1, 2, \dots$, and the latter is captured by $t = \dots, -1, 0, 1, \dots$. There are two types of agents: Type- E agents receive an endowment ω_E at even dates and type- O agents receive an endowment ω_O at odd dates. There are as many type- E agents as type- O agents. We assume that each agent of type- E is paired with a specific type- O agent. Such a pairing occurs every period and two agents never meet more than once. Hence, in order to consume the endowment good of type- E agents, type- O agents must hold money, and vice-versa.⁷

We modify preferences in the following way. Agents do not receive utility from consuming on dates they are endowed with goods. Specifically, type- E agents do not derive utility from consumption in even dates and type- O agents do not derive utility from consumption on odd dates. Agents have access to a technology that allows them to store their endowment from date t to date $t + 1$ at rate x . Type- E (type- O) agents derive utility from consuming their stored endowment on odd (even) dates. Type- E (type- O) agents also derive utility from consuming type- O 's (type- E 's) endowment good at odd (even) dates. Let c^h denote the consumption of an agent's stored endowment good and c^f denote the consumption of the other type's endowment good. If t_0 be an even date, the preferences of type- E agents at date t_0 can be written as

$$\sum_{t=t_0}^{\infty} \beta^{2(t-t_0)-1} \left[u(c_t^h) + u(c_t^f) \right] \quad (10)$$

and the preferences of type- O agents at date t_0 can be written as

$$\sum_{t=t_0}^{\infty} \beta^{2(t-t_0)} \left[u(c_t^h) + u(c_t^f) \right]. \quad (11)$$

If t_0 is an odd date, the preferences of type- E agents at date t_0 is given by equation (11) while the preferences of type- O agents at date t_0 is given by equation (10). When the

⁷As in Townsend (1980) the single-meeting assumption and the absence of a common agent accounts for why credit is not an acceptable means of payment.

economy has an initial date, type- E agents hold money at date zero and get utility only from consuming the endowment good of type- O agents.⁸

The CB chooses z , the rate of growth of the money supply. Money is injected or removed from the economy through lump-sum taxes or subsidies. To facilitate the analogy with the OG setup, we assume the CB levies the tax/subsidy on type- E agents on even dates and on type- O agents at odd dates. Thus, in equilibrium, the CB will levy the tax/subsidy on agents who are holding money.⁹

We assume in this section that β , the agents' discount factor is equal to $1/x$, the inverse of the return on the storage technology. Under this assumption, the Friedman rule equates the return on money with the return on storage and corresponds to a deflation at a rate of β . If we allowed $\beta x > 1$ then agents would want to accumulate assets and the economy would be growing. With $\beta x < 1$, the economy would not grow but we would have to take a stand as to whether the Friedman corresponds to equalizing the return on money and bonds or to a deflation at a rate of β . In either case, our main results would hold: Without an initial date the optimal z is greater than the Friedman rule for either definition. With an initial date the optimal z is no greater than the Friedman rule.

3.1 The economy without an initial date

When there is no initial date, the problem faced by both types of agents is identical. Indeed, the problem of a type- E agent on an even date is the same as the problem of a type- O

⁸To fix ideas, think of the following story. Suppose milk and cheese are perfect substitutes. All agents receive an endowment of milk (some at odd dates, others at even dates). At the time they receive their endowment, agents are not thirsty but can store the milk to turn it into cheese next period at the rate of x units of cheese per unit of milk stored. Next period, agents derive utility from consuming the cheese they have produced as well as the milk that other agents have received as endowment.

⁹Other assumptions about when the taxes/subsidies are being levied do not modify the results. However, they complicate the exposition since, when lump-sum taxes are raised on all agents every period, some agents must hold money for the sole purpose of paying the tax.

agent on an odd date. The problem of any agent can be written recursively as

$$V = \max \beta \left[u(c^h) + u(c^f) \right] + \beta^2 V, \quad (12)$$

subject to

$$m + s \leq (\omega - \tau), \quad (13)$$

$$c^f \leq \frac{m}{1+z}, \quad (14)$$

$$c^h \leq xs. \quad (15)$$

We assume $u(c) = \frac{c^{1-\rho}}{1-\rho}$, where $\rho \in (0, 1)$, and $\gamma = m/(\omega - \tau)$. This implies

$$V = \frac{\beta}{1-\beta^2} \frac{(\omega - \tau)^{1-\rho}}{1-\rho} \left\{ \left[\frac{\gamma}{1+z} \right]^{1-\rho} + [(1-\gamma)x]^{1-\rho} \right\}. \quad (16)$$

Equations (16) and (5) are homomorphic when $\alpha = 1/2$. Thus, the objective function of agents in our modified turnpike economy without an initial date is the same as the objective function of a representative generation in the OG environment. The implication is stated in the following proposition.¹⁰

Proposition 3 *The optimal rate of growth of the money supply in this economy is given by $z = 0$.*

The proof is omitted since this result follows directly from the fact that the environments in this section and the OG environment without an initial date described in section 2 are homomorphic.

¹⁰It is probably useful to offer an interpretation of the homomorphism that exists. Note that there is an equal number of type-E and type-O agents in the Turnpike model. By this assumption, we guarantee that there is always a match at each reststop along the turnpike. The case with $\alpha = 1/2$ is symmetric to assumption of equal-sized groups in the Turnpike setting. In words, $\alpha = 1/2$ says that the number of movers in the OG random relocation economy who want to acquire units of the consumption good with money are exactly the same size, in terms of the fraction of an island's population, as the number of agents that wish to acquire the consumption by exchanging money for goods in the Turnpike model.

The Friedman rule is defined as the rate of growth of the money supply which equates the rate of return of money with the rate of return on storage. Hence, $z^{FR} = 1/x = \beta$. The Friedman rule is not optimal in this economy.

3.2 The economy with an initial date

At the initial date $t = 1$, the problem for type- O agents is not modified compared to the case without initial date. Thus, we can write

$$V^O = \frac{\beta}{1 - \beta^2} \frac{(\omega - \tau)^{1-\rho}}{1 - \rho} \left\{ \left[\frac{\gamma}{1 + z} \right]^{1-\rho} + [(1 - \gamma)x]^{1-\rho} \right\}. \quad (17)$$

The problem for type- E agents, however, is different. Indeed, at date 1 these agents hold money which they can use to buy the endowment good of type- O agents. From date 2 onward, the problem of type- E agents looks like the problem when there is no initial date. It follows that we can write type- E 's problem

$$u(c_0^f) + \beta V^E, \quad (18)$$

where

$$V^E = \beta \left[u(c^h) + u(c^f) \right] + \beta^2 V^E, \quad (19)$$

subject to

$$m + s \leq (\omega - \tau), \quad (20)$$

$$c^f \leq \frac{m}{1 + z}, \quad (21)$$

$$c^h \leq Rs. \quad (22)$$

Date-1 consumption for type- E agents is equal to the real value of the money they possess, M_0/p_1 . Equations (35), (36), and (37), in the appendix, hold in this environment.

We can thus write the problem of type- E agents as

$$u(c_0^f) + \beta V^E = u\left(\frac{\gamma\omega}{(1 + z) - \gamma z}\right) + \frac{\beta^2}{1 - \beta^2} \frac{(\omega - \tau)^{1-\rho}}{1 - \rho} \left\{ \left[\frac{\gamma}{1 + z} \right]^{1-\rho} + [(1 - \gamma)x]^{1-\rho} \right\}. \quad (23)$$

Ex-ante welfare in this economy is given by $W = u(c_0^f) + \beta V^E + V^O$. It is the sum of the utility of type- E and type- O agents. Hence, we have

$$W = u\left(\frac{\gamma\omega}{(1+z) - \gamma z}\right) + \frac{\beta}{1-\beta} \frac{(\omega - \tau)^{1-\rho}}{1-\rho} \left\{ \left[\frac{\gamma}{1+z}\right]^{1-\rho} + [(1-\gamma)x]^{1-\rho} \right\}. \quad (24)$$

Equations (24) and (7) are homomorphic when $\alpha = 1/2$. From proposition 2, the CB would like to set $1+z = \frac{\beta 2^{-\rho}}{1-\beta(1-2^{-\rho})}$. Note that the optimal policy, absent any constraint, would be to set money growth so that the return to money is greater than the return to storage; however this is infeasible. The CB thus sets $1+z = \beta$, the best feasible policy. The implication is that the Friedman rule is the best policy the CB can achieve.

Remark: Note that in the OG framework, β corresponds to the weight the CB puts on a representative generation relative compared to the initial old. In principle, this weight is independent of the discount factor of agents and of the return of the storage technology, x . If β is restricted to be smaller than $1/x$, then a benevolent CB that is restricted to choose the rate of growth of the money supply would implement the Friedman rule. If feasible, the CB would want to implement a slower rate of growth of the money supply.

In the modified turnpike environment, β is the discount factor of agents populating the economy. We have assumed that $\beta = 1/x$ so that the Friedman rule corresponds to a deflation at a rate β and also equate the return on money with the return on storage. Under that assumption, the Friedman rule is the best feasible monetary policy when there is an initial date.

4 Summary and conclusion

In this paper, we constructed an environment populated by infinitely-lived agents which is homomorphic to an OG model. The homomorphism, thus, tells one that there exists a representation of a model economy populated by infinitely-lived agents that is exactly the same as an OG model economy populated by finitely lived agents. As such, the monetary policy results that are derived in the OG setup carry over to the to the infinite-horizon

setting. More specifically, under conditions in which the Friedman rule is not optimal in the OG model economy, the Friedman rule is not optimal in the model economy populated by agents that live infinitely long. In other words, finite lives do not matter. This allows us to establish that the key element for the suboptimality of the Friedman rule in this type of model is the heterogeneity of agents and not the fact that they have finite lives.

Further, our results demonstrate that the existence of an initial date plays an important role. Indeed, in economies without an initial date, we show that all agents face the same problem and agree on what the optimal monetary policy is. However, monetary policy redistributes goods between those holding money at the initial date and those not holding money. For instance, in an OG economy, the initial old generation always prefers a slower rate of growth of the money supply since this increases the value of the money held by members of this generation. In the modified turnpike model, type- E agents prefer a slower rate of growth of the money supply because they hold money at the initial date. In both economies, an increase in the value of money means a larger quantity of goods are affordable.

Thus, our results address two important issues. First, finite lives are not crucial to an analysis of the *ex-ante* optimal monetary policy. Second, whether or not the initial date is treated explicitly matters. Whether or not agents have finite lives, if there is no initial date, the Friedman rule is sub-optimal. If there is an initial date, the Friedman rule is the best feasible policy.

Appendix

A Proof of proposition 1

First define

$$\Omega(z) \equiv \omega - \tau(z) = \omega \frac{1+z}{1+z-z\gamma(z)}, \quad (25)$$

$$\Gamma(z) \equiv \alpha^\rho \left[\frac{\gamma(z)}{1+z} \right]^{1-\rho} + (1-\alpha)^\rho [(1-\gamma(z))x]^{1-\rho}. \quad (26)$$

The objective function of the planner is thus given by

$$W(z) = \frac{\Omega(z)^{1-\rho}}{1-\rho} \Gamma(z) \quad (27)$$

Taking the derivative of $W(z)$ and setting it equal to zero yields

$$\frac{\Omega(z)^{1-\rho} \Gamma(z)}{1+z} \left[\frac{1+z}{\Omega(z)} \frac{\partial \Omega(z)}{\partial z} + \frac{1}{1-\rho} \frac{1+z}{\Gamma(z)} \frac{\partial \Gamma(z)}{\partial z} \right] = 0. \quad (28)$$

It can be verified that

$$\frac{\partial \Gamma(z)}{\partial z} = \alpha^\rho (1-\rho) \frac{1}{1+z} \left(\frac{\gamma(z)}{1+z} \right)^{1-\rho}, \quad (29)$$

$$\frac{\partial \Omega(z)}{\partial z} = \Omega(z) \left[\frac{1}{1+z} - \frac{1-\gamma(z) + z \frac{\partial \gamma(z)}{\partial z}}{1+z+z\gamma(z)} \right], \quad (30)$$

and

$$\frac{\partial \gamma(z)}{\partial z} = \frac{1-\rho}{\rho} \frac{\gamma(z)}{1+z} (\gamma(z) - 1). \quad (31)$$

Using these expressions, it is easy to show that

$$\frac{1}{1-\rho} \frac{1+z}{\Gamma(z)} \frac{\partial \Gamma(z)}{\partial z} = -\gamma(z), \quad (32)$$

and

$$\frac{1+z}{\Omega(z)} \frac{\partial \Omega(z)}{\partial z} = \frac{\gamma(z) \left[1 + \frac{1-\rho}{\rho} z (\gamma(z) - 1) \right]}{1+z-z\gamma(z)}. \quad (33)$$

Substituting these two expressions into equation (28), it can be established that

$$\frac{\partial W(z)}{\partial z} = 0 \Leftrightarrow z = 0. \quad (34)$$

B Proof of proposition 2

Note that, in steady states,

$$-\tau = \frac{M_t - M_{t-1}}{p_t} = m \left(\frac{z}{1+z} \right) \quad (35)$$

$$m = \gamma(\omega - \tau) = \frac{\gamma\omega(1+z)}{(1+z) - \gamma z} \quad (36)$$

and hence,

$$\frac{M_0}{p_1} = \frac{M_1}{p_1} \frac{M_0}{M_1} = m \frac{1}{1+z} = \frac{\gamma\omega}{(1+z) - \gamma z}. \quad (37)$$

Substitute the relevant expressions into the CB's objective function and take the derivative with respect to z , to get

$$\begin{aligned} & \left(\frac{\gamma(z)\omega}{1+z-z\gamma(z)} \right)^{-\rho} \left[\frac{\frac{\partial\gamma(z)}{\partial z}\omega(1+z-z\gamma(z)) - \gamma(z)\omega(1-\gamma(z) - z\frac{\partial\gamma(z)}{\partial z})}{(1+z-z\gamma(z))^2} \right] \\ & + \frac{\beta}{1-\beta} \frac{\Omega(z)^{1-\rho}\Gamma(z)}{1+z} \left[\frac{1+z}{\Omega(z)} \frac{\partial\Omega(z)}{\partial z} + \frac{1}{1-\rho} \frac{1+z}{\Gamma(z)} \frac{\partial\Gamma(z)}{\partial z} \right] = 0 \end{aligned}$$

It can be verified that

$$\frac{1}{1-\rho} \frac{1+z}{\Gamma(z)} \frac{\partial\Gamma(z)}{\partial z} = -\gamma(z)$$

and

$$\frac{1+z}{\Omega(z)} \frac{\partial\Omega(z)}{\partial z} = \frac{\gamma(z) + (1+z)z\frac{\partial\gamma(z)}{\partial z}}{1+z-z\gamma(z)}.$$

After rearranging, we get

$$\begin{aligned} & \left(\frac{\gamma(z)\omega}{1+z-z\gamma(z)} \right)^{-\rho} \omega \left[\frac{\frac{\partial\gamma(z)}{\partial z}(1+z) - \gamma(z)(1-\gamma(z))}{(1+z-z\gamma(z))^2} \right] \\ & + \frac{\beta}{1-\beta} \frac{\Gamma(z)}{1+z} \frac{(\omega(1+z))^{1-\rho}}{(1+z-z\gamma(z))^{1-\rho}} \frac{z \left[\frac{\partial\gamma(z)}{\partial z}(1+z) - \gamma(z)(1-\gamma(z)) \right]}{(1+z-z\gamma(z))} = 0, \end{aligned}$$

which simplifies to

$$\gamma(z)^{-\rho} + \frac{\beta}{1-\beta} \Gamma(z) (\omega(1+z))^{-\rho} z = 0.$$

Note

$$\begin{aligned}\Gamma(z) &= \alpha^\rho \left(\frac{\gamma(z)}{1+z} \right)^{1-\rho} + (1-\alpha)^\rho [(1-\gamma(z))x]^{1-\rho} \\ &= \left(\frac{\alpha}{\gamma(z)} \right)^\rho \frac{\gamma(z)}{(1+z)^{1-\rho}} + \left(\frac{1-\alpha}{1-\gamma(z)} \right)^\rho (1-\gamma(z)) x^{1-\rho}.\end{aligned}$$

From the bank's maximization we have, $\alpha^\rho \frac{1}{1+z} \left(\frac{\gamma(z)}{1+z} \right)^{-\rho} - \left(\frac{1-\alpha}{1-\gamma(z)} \right)^\rho x^{1-\rho} = 0$, so that

$$\Gamma(z) = \left(\frac{\alpha}{\gamma(z)} \right)^\rho \frac{1}{(1+z)^{1-\rho}}.$$

Substitution for Γ in the above expression, we get

$$\gamma^{-\rho} = -\frac{\beta}{1-\beta} \alpha^\rho \gamma^{-\rho} (1+z)^{-1} z.$$

With a little algebra this becomes

$$1+z = 1 - \frac{1-\beta}{1-\beta(1-\alpha^\rho)}.$$

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