

Customization with Vertically Differentiated Products*

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Abstract

We study an asymmetric duopoly market in which the firms' products are initially differentiated in both variety and quality. Each consumer has a most preferred variety and a quality valuation. Customization provides ideal varieties for consumers but has no effect on product qualities. The firms first choose whether to customize their products, then engage in price competition. For the customization stage we consider two different games: the simultaneous-move game and the endogenous-timing game. In the latter, whether customization choices are made simultaneously or sequentially is endogenously determined. We show that both quality and the timing of customization choices play important roles in determining the equilibrium outcome. Customization occurs only if the quality difference is sufficiently large. Endogenous timing sometimes enables the firms to achieve an outcome that is Pareto superior to that if they were to make their customization choices simultaneously. Although the higher quality firm is more likely to customize, endogenous timing sometimes enables the lower quality firm to obtain an advantage that it would not have in the simultaneous-move game.

Key words: customization, horizontal differentiation, vertical differentiation, endogenous timing.

JEL codes: D43, L13, C72.

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1 Introduction

Customization is a flexible technology designed to produce individually tailored products without significantly compromising cost efficiency. Advances in Internet-based information technologies and improvements in manufacturing flexibility have made customization a reality in many product categories. For example, Dell builds to order notebook and desktop computers; NikeID.com allows consumers to create their most preferred athletic pair of shoes; Timbuk2 customizes messenger bags and backpacks; apparel vendor LandsEnd.com offers custom-crafted pants and shirts.

Most of the existing theoretical literature on customization adopt one-dimensional horizontal differentiation settings (e.g., Dewan, Jing, and Seidmann 2003, Syam and Kumar 2006, Alexandrov 2008, and Mendelson and Parlaktürk 2008). Customization enables consumers to get their ideal products represented by their locations in the product attribute space. Firms are symmetrical and make symmetric choices in equilibrium.¹ Even though many important aspects of customization are captured by these studies, important issues have yet to be examined. Casual empiricism indicates that (i) some firms customize, some do not; (ii) firms may not make their customization choices at the same time; and (iii) higher quality firms are more likely to offer customization. The goal of the present paper is to incorporate these observations into product customization competition.

We study an industry in which products are characterized by variety and quality. Variety is a horizontal attribute and quality is a vertical attribute.² Consumer preferences are heterogenous in two dimensions. In particular, each consumer has a most preferred variety and a quality valuation. There are two firms that initially produce standard products located at the end points of the variety space. The firms are asymmetrical due to having different qualities. Customization provides ideal varieties for consumers but has no effect on product qualities. The model has two stages. The first is the customization stage, in which the firms select their product types (standard or customized). In the second stage – the pricing stage – the firms engage in price competition.

We consider two different games to model the customization stage. In one game the firms simultaneously decide whether to customize their products. In the other game the customization stage unfolds in two periods. Each firm either selects its product type in period 1 or postpones this decision to period 2. Thus, whether the customization choices are made simultaneously or sequentially is endogenously determined. While the first game focuses on issues related to observations (i) and (iii) above, the second game addresses all three observations. We will call the first game the simultaneous-move game and the second the endogenous-timing game.

Industrial organization economists have long been interested in whether firms choose their prices and/or quantities simultaneously or sequentially. Hamilton and Slutsky (1990, 1993) were the first to introduce a model in which the determination of simultaneity versus sequentiality of moves is endogenous.³ The structure of our endogenous-timing game follows Hamilton and Slutsky (1993),

¹An exception is Mendelson and Parlaktürk (2008) in which asymmetric firms are considered.

²Our basic model is based on the literature that combines horizontal and vertical differentiation, e.g., Economides (1989) and Neven and Thisse (1990).

³Many subsequent studies have examined endogenous timing and related issues in standard price and/or quantity duopoly games, e.g., Matsumura (1999) and Amir and Stepanova (2006).

in which the basic game is given by a 2×2 matrix and the extended game has two periods. This is the simplest possible setup to model firms' flexibility in choosing when to move.

The paper on customization that is closest to ours is Syam, Ruan, and Hess (2005). In both papers the consumer space is two-dimensional. Syam et al. (2005) endow products with two horizontal attributes, for which consumers have heterogeneous preferences. The firms are initially maximally differentiated with respect to both attributes. They first simultaneously choose whether to customize both, one, or none of the attributes, then compete in prices. The key difference between Syam et al. (2005) and our study is that they work with ex ante symmetric firms and examine how the possibility of customizing multiple attributes affect customization choices, whereas we work with asymmetric firms and focus on the roles of quality difference and endogenous timing in customization competition.

Another closely related paper is Bernhardt, Liu, and Serfes (2007), in which ex ante symmetric firms first acquire information about consumers and then customize their products as best as they can to match consumer needs. Similar to our paper, consumer preferences are two-dimensional, corresponding to two attributes of the product, and the second attribute – brand loyalty – cannot be customized. There are two main differences between Bernhardt et al. (2007) and our study. First, Bernhardt et al. (2007) emphasize the cost side of customization, whereas our focus is shifted towards the strategic effects of customization. Second, brand loyalty is a horizontal attribute, not vertical as quality in our paper is.

A number of papers have studied customization using a one-dimensional consumer space. Dewan et al. (2003) assume that customizing firms price discriminate. Syam and Kumar (2006) examine the role of standard products in customization competition. Alexandrov (2008) extends Salop's (1979) model in which firms can offer interval-long adjustable "fat" products. In a dynamic setting Chen (2006) studies two marketing innovations, one of which is essentially a form of product customization. While these studies assume symmetric firms, in Mendelson and Parlaktürk (2008) one firm has a margin advantage (higher difference between reservation price and unit cost) over the other.

The present paper as well as Syam et al. (2005) model customization as zero-one decisions, so that all customers of a customizing firm get their most preferred varieties. In contrast, all the other papers mentioned above treat customization as continuous choices. Both approaches match aspects of reality and have their advantages. With zero-one decisions, more attention can be devoted to the strategic effects of customization. With continuous customization choices, one can focus on how efficiency considerations determine the range of customization.

Our equilibrium analysis shows that quality difference and endogenous timing play important roles in determining the equilibrium outcome. In particular, no firm will customize if the quality difference is small, regardless of the fixed cost of customization. Intuitively, customization by one or both firms makes their products less differentiated, thus intensifying price competition. The smaller is the quality difference, the tougher is price competition. For sufficiently large quality differences, customization by one or both firms may occur. Because the high quality firm benefits more from customization than the low quality firm, the high quality firm is more likely to customize.

We show that the endogeneity of timing in the customization stage is an additional strategic tool

for the firms. It sometimes enables the firms to achieve an outcome that is Pareto superior to that if they were to make their customization decisions simultaneously. While the low quality firm never customizes alone in the simultaneous-move game, in the endogenous-timing game it may obtain an advantage by becoming the first and only firm to customize.

The rest of the paper is organized as follows. In the next section we introduce the model. In Section 3 the pricing stage is analyzed. In Sections 4 and 5 we study the firms' customization choices in the simultaneous-move game and the endogenous-timing game, respectively. Concluding remarks are provided in Section 6. Proofs of all lemmas and propositions are relegated to the Appendix.

2 The Model

Consider a market in which each product i is characterized by its variety $x_i \in [0, 1]$ and its quality $q_i \geq 0$. The first characteristic corresponds to horizontal differentiation and the second to vertical differentiation. Consumers are heterogenous in two dimensions. Each consumer has a most preferred variety $x \in [0, 1]$ and a quality valuation $y \in [0, 1]$. A consumer of type (x, y) derives the following utility from buying one unit of product i :

$$v + q_i y - t|x - x_i| - p_i,$$

where v is a positive constant, t is a preference parameter, and p_i is the price of product i . Consumers as represented by (x, y) are uniformly distributed over the unit square $[0, 1] \times [0, 1]$ with a total mass of 1. We assume that v is large enough for all consumers to find a product that yields positive payoff in equilibrium.

There are two firms, A and B, operating with zero marginal costs of production. Initially, firm A offers a single (standard) product of quality q_A and variety $x_A = 0$, whereas firm B offers a single product of quality $q_B > q_A$ and variety $x_B = 1$. That is, firm B is the higher quality firm and the two firms have maximum variety differentiation.

We will normalize t to 1. This amounts to a monotonic transformation of preferences. The utilities of a consumer of type (x, y) from buying firm A's and firm B's standard product are

$$v + q_A y - x - p_A \tag{1}$$

and

$$v + q_B y - (1 - x) - p_B, \tag{2}$$

respectively.

Investing $K \geq 0$ into product-customization technology allows a firm to produce a product that exactly matches a given consumer's preferred variety. The utilities of type (x, y) from buying firm A's and firm B's customized product are

$$v + q_A y - p_A \tag{3}$$

and

$$v + q_B y - p_B. \tag{4}$$

We consider two different games: the simultaneous-move game and the endogenous-timing game. Both games have two stages, a customization stage followed by a pricing stage. The difference between the two games is in the customization stage. In the simultaneous-move game, the firms choose simultaneously whether to customize their products. In the the endogenous-timing game the customization stage involves two periods. Each firm selects its product type (standard or customized) in period 1 or postpones this decision to period 2. Thus, whether the product type choices are made simultaneously or sequentially is endogenously determined. After the product types are chosen, the firms enter the pricing stage, in which they choose prices simultaneously. Consumers decide which product to purchase, and the profits are realized. We adopt subgame perfect Nash equilibrium and use backward induction to solve the two games.

3 Analysis of the Pricing Stage

The customization stage leads to four possible outcomes: both firms choose standard products (SS), only firm A customizes (CS), only firm B customizes (SC), and both firms customize (CC). We next investigate the firms' pricing decisions following each of these outcomes.

3.1 Equilibrium Prices Following SS

Suppose both firms selected standard products. Utilities from firm A's and firm B's standard products are given by (1) and (2). Therefore, for a given level of quality valuation y , the marginal consumer type in terms of variety x is

$$\hat{x}(y) = \frac{1}{2} (1 - qy + p_B - p_A), \tag{5}$$

where

$$q \equiv q_B - q_A$$

denotes the quality difference between the firms' products. For any $y \in [0, 1]$, consumers in the interval $x \in [0, \hat{x}(y)]$ will purchase from firm A, whereas those with $x \in (\hat{x}(y), 1]$ will purchase from firm B. There are four possible positions for the indifference line (5), as illustrated in Figure 1. The slope of the indifference line equals $-2/q$. An increase in q makes the line flatter. An increase in p_B (and/or decrease in p_A) shifts the line to the right, thereby reducing the market size of firm B.

Let $D_A(p_A, p_B)$ and $D_B(p_A, p_B)$ denote the demand functions of firms A and B. The expressions for these functions depend on the position of the indifference line.⁴ The firms choose simultaneously p_A and p_B to maximize their profits,

$$\Pi_A(p_A, p_B) = D_A(p_A, p_B)p_A$$

⁴The details are provided in the Appendix.

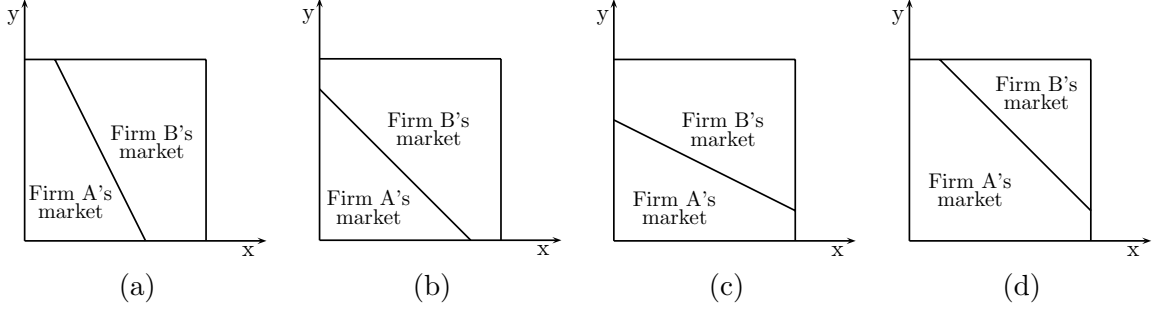


Figure 1: Indifference line and market areas

and

$$\Pi_B(p_A, p_B) = D_B(p_A, p_B)p_B.$$

Lemma 1 (Equilibrium prices and profits following SS). *Suppose both firms selected standard products in the customization stage, then the equilibrium prices and profits in the pricing stage are as follows.*

(i) If $q \leq 3/2$,

$$\begin{cases} p_A^{SS} = 1 - \frac{1}{6}q \\ p_B^{SS} = 1 + \frac{1}{6}q \end{cases} \text{ and } \begin{cases} \Pi_A^{SS} = \frac{1}{2} \left(1 - \frac{1}{6}q\right)^2 \\ \Pi_B^{SS} = \frac{1}{2} \left(1 + \frac{1}{6}q\right)^2 \end{cases}$$

(ii) If $q \in (3/2, 3]$,

$$\begin{cases} p_A^{SS} = \frac{1}{8} (1 + \sqrt{1 + 16q}) \\ p_B^{SS} = \frac{1}{8} (-5 + 3\sqrt{1 + 16q}) \end{cases} \text{ and } \begin{cases} \Pi_A^{SS} = \frac{1}{q} \left(\frac{1 + \sqrt{1 + 16q}}{8}\right)^3 \\ \Pi_B^{SS} = \left(1 - \frac{1}{q} \left(\frac{1 + \sqrt{1 + 16q}}{8}\right)^2\right) \frac{-5 + 3\sqrt{1 + 16q}}{8} \end{cases}$$

(iii) If $q > 3$,

$$\begin{cases} p_A^{SS} = \frac{1}{3}q \\ p_B^{SS} = \frac{2}{3}q \end{cases} \text{ and } \begin{cases} \Pi_A^{SS} = \frac{1}{9}q \\ \Pi_B^{SS} = \frac{4}{9}q \end{cases}$$

It is easy to verify that the corresponding prices and profits in parts (i) and (ii) are equal when evaluated at $q = 3/2$. Similarly, the prices and profits in parts (ii) and (iii) are equal at $q = 3$. Therefore, the equilibrium prices and profits vary continuously when q changes.

Case (i) corresponds to Figure 1(a) in which q is small. In equilibrium, both firms serve consumers with all quality valuations, and each firm attracts consumers closer to its position on the variety interval. Case (ii) corresponds to Figure 1(b). Firm A attracts only consumers who are close to its variety position and have low quality valuations (i.e., small x 's and small y 's). Firm B uses its quality advantage to capture all the other consumers. Case (iii) corresponds to Figure 1(c) in which quality difference q is large. In this case, the low quality firm A competes aggressively and the high quality firm B does better setting a high price and exploiting consumers with high quality valuations. In equilibrium, firm A serves consumers of all variety preferences, and so does firm B.

Firm A attracts consumers with low quality valuations and firm B attracts consumers with high quality valuations.

It is worthwhile to note the intuitive outcome implied by Lemma 1 that in all three cases firm B sets a higher price, serves a larger market area, and earns a higher profit than firm A. Furthermore, Figure 1(d) does not arise in equilibrium.

3.2 Equilibrium Prices Following CS

Suppose firm A selected a customized product and firm B selected a standard product. Utilities from firm A's customized product and firm B's standard product are given by (3) and (2). Therefore, for a given y , the marginal consumer type in terms of x is

$$\hat{x}(y) = 1 - qy + p_B - p_A, \quad (6)$$

from which we can calculate the demand and profit functions for the firms. Lemma 2 presents the equilibrium prices and profits following outcome CS.

Lemma 2 (Equilibrium prices and profits following CS). *Suppose firm A selected a customized product and firm B selected a standard product in the customization stage, then the equilibrium prices and profits in the pricing stage are as follows.*

(i) If $q \leq 1$,

$$\begin{cases} p_A^{CS} = \frac{2}{3} - \frac{1}{6}q \\ p_B^{CS} = \frac{1}{3} + \frac{1}{6}q \end{cases} \quad \text{and} \quad \begin{cases} \Pi_A^{CS} = \left(\frac{2}{3} - \frac{1}{6}q\right)^2 \\ \Pi_B^{CS} = \left(\frac{1}{3} + \frac{1}{6}q\right)^2 \end{cases}$$

(ii) If $q > 1$,

$$\begin{cases} p_A^{CS} = \frac{1}{3}q + \frac{1}{6} \\ p_B^{CS} = \frac{2}{3}q - \frac{1}{6} \end{cases} \quad \text{and} \quad \begin{cases} \Pi_A^{CS} = \frac{1}{q} \left(\frac{1}{3}q + \frac{1}{6}\right)^2 \\ \Pi_B^{CS} = \frac{1}{q} \left(\frac{2}{3}q - \frac{1}{6}\right)^2 \end{cases}$$

Note that the critical values for the cases in this lemma are different from those in Lemma 1, and that the pricing equilibrium following CS leads to only two possible positions of the indifference line. This is due to the fact that the slope of the indifference line (6) is $-1/q$, which is different from that of (5).

Case (i) of Lemma 2 corresponds to Figure 1(a) in which quality difference q is small. Customization enables firm A to overcome its quality disadvantage. Firm A's equilibrium price, market size, and profit are higher than those of firm B. In equilibrium, both firms serve consumers with all quality valuations, and each firm attracts consumers closer to its position on the variety interval. To explain why firm B still attracts consumers with low quality valuations, it suffices to consider consumers with $y = 0$. Such consumers do not gain any extra utility buying from the high quality firm, yet some of them are attracted by firm B because of its low price. Case (ii) corresponds to Figure 1(c) in which quality difference q is large. In this case, customization does not overcome the quality disadvantage of firm A. In equilibrium, firm A's price, market size, and profit are lower than those of firm B. Firm A serves consumers of all variety preferences and so does firm B. Firm

A attracts consumers with low quality valuations and firm B attracts consumers with high quality valuations.

3.3 Equilibrium Prices Following SC

Suppose firm A selected a standard product and firm B selected a customized product. Utilities from firm A's standard product and firm B's customized product are given by (1) and (4). Therefore, for a given y , the marginal consumer type in terms of x is

$$\hat{x}(y) = -qy + p_B - p_A. \quad (7)$$

The next lemma presents the equilibrium prices and profits following outcome SC.

Lemma 3 (Equilibrium prices and profits following SC). *Suppose firm A selected a standard product and firm B selected a customized product in the customization stage, then the equilibrium prices and profits in the pricing stage are as follows.*

(i) If $q \leq 1/2$,

$$\begin{cases} p_A^{SC} = \frac{1}{3} - \frac{1}{6}q \\ p_B^{SC} = \frac{2}{3} + \frac{1}{6}q \end{cases} \quad \text{and} \quad \begin{cases} \Pi_A^{SC} = \left(\frac{1}{3} - \frac{1}{6}q\right)^2 \\ \Pi_B^{SC} = \left(\frac{2}{3} + \frac{1}{6}q\right)^2 \end{cases}$$

(ii) If $q \in (1/2, 2]$,

$$\begin{cases} p_A^{SC} = \frac{1}{4}\sqrt{2q} \\ p_B^{SC} = \frac{3}{4}\sqrt{2q} \end{cases} \quad \text{and} \quad \begin{cases} \Pi_A^{SC} = \frac{1}{16}\sqrt{2q} \\ \Pi_B^{SC} = \frac{9}{16}\sqrt{2q} \end{cases}$$

(iii) If $q > 2$,

$$\begin{cases} p_A^{SC} = \frac{1}{3}q - \frac{1}{6} \\ p_B^{SC} = \frac{2}{3}q + \frac{1}{6} \end{cases} \quad \text{and} \quad \begin{cases} \Pi_A^{SC} = \frac{1}{q} \left(\frac{1}{3}q - \frac{1}{6}\right)^2 \\ \Pi_B^{SC} = \frac{1}{q} \left(\frac{2}{3}q + \frac{1}{6}\right)^2 \end{cases}$$

Case (i) corresponds to Figure 1(a), case (ii) to 1(b), and case (iii) to 1(c). Firm B's quality advantage is reinforced by customization, pushing the critical values lower compared to those in Lemma 1.

3.4 Equilibrium Prices Following CC

Suppose both firms selected customized products. Utilities from firm A's and firm B's customized products are given by (3) and (4). Therefore, consumers with $y \leq (p_B - p_A)/q$ will purchase from firm A, those with $y > (p_B - p_A)/q$ will purchase from firm B. The firms' profit functions are

$$\Pi_A(p_A, p_B) = \frac{1}{q} (p_B - p_A) p_A \quad \text{and} \quad \Pi_B(p_A, p_B) = \frac{1}{q} (q + p_A - p_B) p_B.$$

The profit maximizing first-order conditions

$$\begin{cases} p_B - 2p_A = 0 \\ q + p_A - 2p_B = 0 \end{cases}$$

lead immediately to the following lemma.

Lemma 4 (Equilibrium prices and profits following CC). *Suppose both firms selected customized products in the customization stage, then the equilibrium prices and profits in the pricing stage are given by:*

$$\left\{ \begin{array}{l} p_A^{CC} = \frac{1}{3}q \\ p_B^{CC} = \frac{2}{3}q \end{array} \right. \text{ and } \left\{ \begin{array}{l} \Pi_A^{CC} = \frac{1}{9}q \\ \Pi_B^{CC} = \frac{4}{9}q \end{array} \right.$$

In equilibrium the indifference line is horizontal at $1/3$ from the bottom side of the unit square. That is, firm A serves all consumers with quality valuations less than $1/3$, and firm B serves the rest. The result here is the same as in the standard model of vertical differentiation.

Having derived the profit functions for the pricing stage, we move one step back to study the customization stage. In the next section we study the simultaneous-move game, in Section 5 the endogenous-timing game.

4 Simultaneous-Move Game

In the customization stage of the simultaneous-move game, each firm has two possible strategies: choosing a standard product (S) or choosing a customized product (C). This stage is represented by the following matrix.

		Firm B	
		S	C
Firm A	S	Π_A^{SS}, Π_B^{SS}	$\Pi_A^{SC}, \Pi_B^{SC} - K$
	C	$\Pi_A^{CS} - K, \Pi_B^{CS}$	$\Pi_A^{CC} - K, \Pi_B^{CC} - K$

It follows that

$$\left. \begin{array}{l} (S,S) \\ (C,S) \\ (S,C) \\ (C,C) \end{array} \right\} \text{ is a Nash equilibrium if } \left\{ \begin{array}{l} K \geq \max\{c_1, r_1\} \\ K \in [r_2, c_1] \\ K \in [c_2, r_1] \\ K \leq \min\{c_2, r_2\} \end{array} \right.$$

where

$$c_1 \equiv \Pi_A^{CS} - \Pi_A^{SS} \quad \text{and} \quad c_2 \equiv \Pi_A^{CC} - \Pi_A^{SC}$$

denote firm A's change in gross profit in the two columns, and

$$r_1 \equiv \Pi_B^{SC} - \Pi_B^{SS} \quad \text{and} \quad r_2 \equiv \Pi_B^{CC} - \Pi_B^{CS}$$

denote firm B's change in gross profit in the two rows.

Lemma 5 (Relative gains from customization). *For any value of q , $\max\{c_1, c_2\} < \min\{r_1, r_2\}$.*

Detailed expressions for c_1 , c_2 , r_1 , and r_2 as functions of q are provided in the Appendix. This lemma stipulates that the low quality firm A always gains less from customization than the high

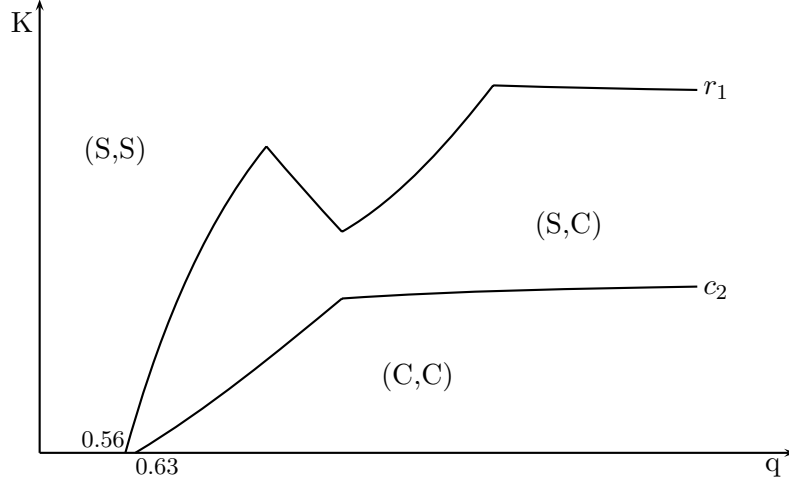


Figure 2: Equilibrium customization choices in the simultaneous-move game

quality firm B.

It follows from Lemma 5 that the simultaneous-move game has a unique Nash equilibrium except for $K = c_2$ and $K = r_1$. For ease of presentation without affecting the results, we will select (C,C) as the Nash equilibrium at $K = c_2$ and (S,C) at $K = r_1$. Accordingly,

$$\left. \begin{array}{l} (S,S) \\ (S,C) \\ (C,C) \end{array} \right\} \text{ is the Nash equilibrium if } \left\{ \begin{array}{l} K > r_1 \\ K \in (c_2, r_1] \\ K \leq c_2 \end{array} \right.$$

Because K is non-negative, the above discussion implies that the signs of c_2 and r_1 are crucial for equilibrium analysis. Specifically, when both c_2 and r_1 are negative (S,S) is the Nash equilibrium for any K . When $c_2 < 0$ and $r_1 > 0$, either (S,C) or (S,S) is the Nash equilibrium, depending on K . When both c_2 and r_1 are positive, the Nash equilibrium can be (C,C), (S,C), or (S,S).

The next proposition summarizes our main results on customization choices in the simultaneous-move game.

Proposition 1 (Equilibrium customization choices in the simultaneous-move game). *The following holds for the firms' equilibrium customization choices in the simultaneous-move game.*

- (i) *If $q \leq 0.56$ then the Nash equilibrium is (S,S) for any value of K .*
- (ii) *If $q \in (0.56, 0.63]$ then the Nash equilibrium is (S,C) for $K \leq r_1$ and (S,S) for $K > r_1$.*
- (iii) *If $q > 0.63$ then the Nash equilibrium is (C,C) for $K \leq c_2$, (S,C) for $K \in (c_2, r_1]$, and (S,S) for $K > r_1$.*

Figure 2 plots c_2 and r_1 as functions of q .⁵ Depending on the values for parameters q and K , either

⁵The number 0.56 is an approximate solution to $81\sqrt{2q} = 2(6+q)^2$. The number 0.63 is an approximate value of $81/126$.

both firms customize, only the high quality firm customizes, or no firm customizes. Moreover, the appearance of this sequence of outcomes is monotone in the fixed cost of customization. Specifically, starting at the equilibrium when $K = 0$ we always move down the sequence (C,C)–(S,C)–(S,S) as K increases. Proposition 1 also implies that even if customization is costless, the firms might not customize in equilibrium. This happens when the quality difference q is small (≤ 0.56).

It is worth noting that while c_2 is an increasing function of q , r_1 is not. As a result, for fixed K , the equilibrium does not follow a certain sequence as q changes. This can be seen from Figure 2. For example, as q increases, the equilibrium changes from (S,S) to (S,C) to (C,C) for small values of K , and from (S,S) to (S,C) to (S,S) to (S,C) for a range of values of K .

To highlight the effect of q on the equilibrium customization choices, consider $K = 0$. Customization by one or both firms makes the rivals “closer” to each other, thus intensifying price competition. The smaller is the quality difference, the tougher is price competition. In the extreme (hypothetical) case in which $q = 0$ and both firms customize, price competition results in the Bertrand outcome. This intuition is behind the findings of Proposition 1. When q is small, the firms do not customize their products in order to avoid a price war. When q is large, the firms customize to take advantage of consumers’ desires for ideal varieties. The intermediate case involves customization by one of the firms (firm B).

5 Endogenous-Timing Game

In this section we analyze the firms’ customization decisions in the endogenous-timing game. Figure 3 depicts its extensive form. In period 1 the firms choose simultaneously between selecting their product types (standard or customized) this period or wait until period 2. We use S, C, and W to denote these three choices. If both firms selected their product types in period 1, the game proceeds to the second stage. If one firm chose to wait in period 1, it selects its product type in period 2. If both firms chose to wait in period 1, they simultaneously select their product types in period 2.

5.1 Preliminary Analysis

We apply directly the results in Hamilton and Slutsky (1993). Because the simultaneous-move game has a unique Nash equilibrium in pure strategies, either the Nash equilibrium is in dominant strategies, or it involves one dominant strategy.⁶ When the Nash equilibrium is in dominant strategies, the order of product type selection in the endogenous-timing game does not matter. This occurs when K is neither in between c_1 and c_2 nor in between r_1 and r_2 . There are multiple subgame perfect equilibria yielding a unique outcome that coincides with the Nash equilibrium of the simultaneous-move game. More specifically, the following holds.

- If $K \leq \min\{c_1, c_2\}$, then both firms select customized products.

⁶See Lemma I of Hamilton and Slutsky (1993).

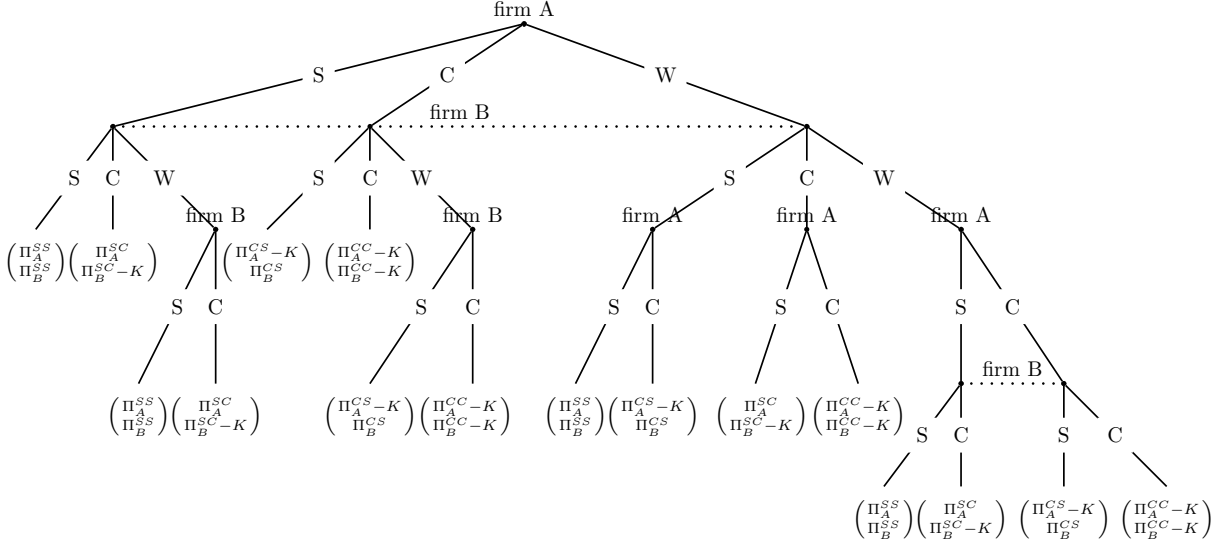


Figure 3: Customization stage of the endogenous-timing game

- If $K \in (\max\{c_1, c_2\}, \min\{r_1, r_2\}]$, then firm A selects a standard product and firm B selects a customized product.
- If $K > \max\{r_1, r_2\}$, then both firms select standard products.

When the Nash equilibrium of the simultaneous-move game involves only one dominant strategy, the endogenous-timing game may lead to different customization choices.⁷ This occurs when K is either in between c_1 and c_2 or in between r_1 and r_2 . Hence, it is left to investigate the four regions highlighted in Figure 4. As in Hamilton and Slutsky (1993), we will focus on equilibria in undominated strategies.

The following two lemmas will be helpful for our subsequent analysis.

Lemma 6 (Outcome SS versus outcome CC). $\Pi_A^{SS} \geq \Pi_A^{CC}$ and $\Pi_B^{SS} \geq \Pi_B^{CC}$ for any value of q .

This lemma stipulates that both firms fare better when they produce standard products than when they customize, even if the fixed cost of customization is zero. Obviously, this is due to the reduced competition in outcome SS compared to CC.

Lemma 7 (Outcome SC versus outcome CS). $\Pi_A^{SC} + \Pi_B^{SC} > \Pi_A^{CS} + \Pi_B^{CS}$ for any value of q .

Intuitively, customization makes the low quality firm more aggressive than it does the high quality firm. Hence, price competition following outcome CS is more fierce compared to that following SC. This is responsible for the result in Lemma 7 that the industry profit is higher when the high quality firm customizes than when the low quality firm customizes.

⁷See Theorem III of Hamilton and Slutsky (1993).

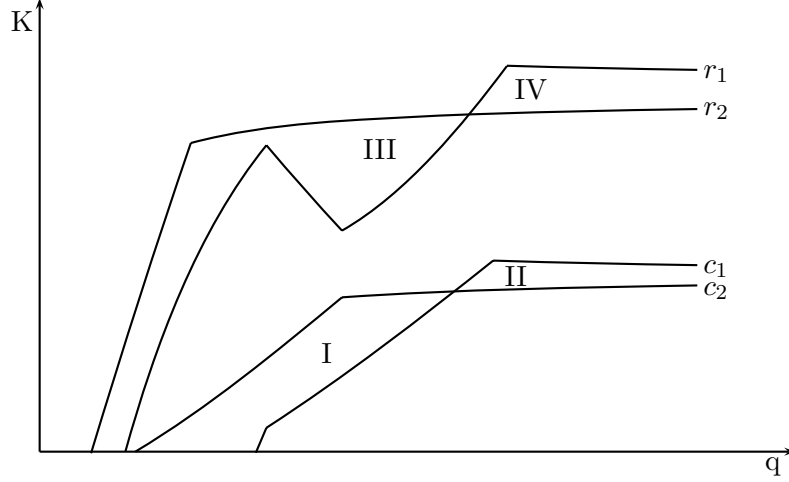


Figure 4: Regions in which NE of the simultaneous-move game has one dominant strategy

5.2 Region I

Here, we have $K \in (c_1, c_2]$. The Nash equilibrium of the simultaneous-move game is (C,C) and only firm B has a dominant strategy. Applying backward induction to the endogenous-timing game leads to the following 3×3 payoff matrix for the period 1 strategies:

		Firm B			
		S	C	W	
Firm A	S	Π_A^{SS}, Π_B^{SS}	$\Pi_A^{SC}, \Pi_B^{SC} - K$	$\Pi_A^{SC}, \Pi_B^{SC} - K$	(8)
	C	$\Pi_A^{CS} - K, \Pi_B^{CS}$	$\Pi_A^{CC} - K, \Pi_B^{CC} - K$	$\Pi_A^{CC} - K, \Pi_B^{CC} - K$	
	W	Π_A^{SS}, Π_B^{SS}	$\Pi_A^{CC} - K, \Pi_B^{CC} - K$	$\Pi_A^{CC} - K, \Pi_B^{CC} - K$	

This matrix deserves a discussion. The upper left 2×2 submatrix is obvious. Cell (W,W) reflects the fact that (C,C) is the Nash equilibrium of the simultaneous-move game. If in period 1 firm A selects its product type and firm B waits, then in period 2 firm B's best response is to customize. This is reflected in the off-diagonal cells of the last column, (S,W) and (C,W). If in period 1 firm B selects its product type and firm A waits, then in period 2 firm A will match firm B's selection (cells (W,S) and (W,C) in the matrix).

Because $K \in (c_1, c_2]$,

$$\Pi_A^{SS} > \Pi_A^{CS} - K \quad \text{and} \quad \Pi_A^{CC} - K \geq \Pi_A^{SC},$$

W weakly dominates S and C for firm A. From Lemma 6,

$$\Pi_B^{SS} > \Pi_B^{CC} - K$$

holds for any q and K in region I. It follows that firm B will select a standard product against

waiting by firm A. The unique Nash equilibrium for matrix (8) in undominated strategies is (W,S). The outcome corresponding to this equilibrium is SS.⁸ Observe that, by Lemma 6, this outcome is Pareto superior to that in the simultaneous-move game.

5.3 Region II

In this region, $K \in (c_2, c_1]$. The Nash equilibrium of the simultaneous-move game is (S,C) and only firm B has a dominant strategy. Applying backward induction leads to the following game matrix for period 1:

		Firm B		
		S	C	W
Firm A	S	Π_A^{SS}, Π_B^{SS}	$\Pi_A^{SC}, \Pi_B^{SC} - K$	$\Pi_A^{SC}, \Pi_B^{SC} - K$
	C	$\Pi_A^{CS} - K, \Pi_B^{CS}$	$\Pi_A^{CC} - K, \Pi_B^{CC} - K$	$\Pi_A^{CC} - K, \Pi_B^{CC} - K$
	W	$\Pi_A^{CS} - K, \Pi_B^{CS}$	$\Pi_A^{SC}, \Pi_B^{SC} - K$	$\Pi_A^{SC}, \Pi_B^{SC} - K$

(9)

This matrix differs from matrix (8) in the last row. If in period 1 firm B selects its product type and firm A waits, then in period 2 firm A's best reply will be to make a different selection from firm B's. This is reflected in cells (W,S) and (W,C). Cell (W,W) is due to the fact that (S,C) is the Nash equilibrium of the simultaneous-move game.

It follows from $K \in (c_2, c_1]$ that waiting is a weakly dominant strategy for firm A. We show in the Appendix that for any q and K in region II

$$\Pi_B^{SC} - K > \Pi_B^{CS}. \quad (10)$$

Hence, firm B will choose either C or W against firm A's waiting. The Nash equilibria for matrix (9) in undominated strategies are (W,C) and (W,W). Both yield outcome SC, the same as in the simultaneous-move game.

5.4 Region III

In this region, $K \in (r_1, r_2]$. The Nash equilibrium of the simultaneous-move game is (S,S) and only firm A has a dominant strategy. Applying backward induction leads to the following matrix for period 1:

		Firm B		
		S	C	W
Firm A	S	Π_A^{SS}, Π_B^{SS}	$\Pi_A^{SC}, \Pi_B^{SC} - K$	Π_A^{SS}, Π_B^{SS}
	C	$\Pi_A^{CS} - K, \Pi_B^{CS}$	$\Pi_A^{CC} - K, \Pi_B^{CC} - K$	$\Pi_A^{CC} - K, \Pi_B^{CC} - K$
	W	Π_A^{SS}, Π_B^{SS}	$\Pi_A^{SC}, \Pi_B^{SC} - K$	Π_A^{SS}, Π_B^{SS}

(11)

⁸Matrix (8) has two Nash equilibria in weakly dominated strategies (C,C) and (C,W), with outcome CC.

Here, firm B has a weakly dominant strategy to wait. Because

$$\Pi_A^{SS} > \Pi_A^{CC} - K$$

holds for any q and K in region III by Lemma 6, firm A will choose either S or W against waiting by firm B. The Nash equilibria in undominated strategies for matrix (11) are (S,W) and (W,W). Both yield outcome SS, the same as in the simultaneous-move game.

5.5 Region IV

Region IV turns out to be most interesting. In this region, $K \in (r_2, r_1]$. The Nash equilibrium of the simultaneous-move game is (S,C) and only firm A has a dominant strategy. Applying backward induction leads to the following matrix for period 1:

		Firm B		
		S	C	W
Firm A	S	Π_A^{SS}, Π_B^{SS}	$\Pi_A^{SC}, \Pi_B^{SC} - K$	$\Pi_A^{SC}, \Pi_B^{SC} - K$
	C	$\Pi_A^{CS} - K, \Pi_B^{CS}$	$\Pi_A^{CC} - K, \Pi_B^{CC} - K$	$\Pi_A^{CS} - K, \Pi_B^{CS}$
	W	Π_A^{SS}, Π_B^{SS}	$\Pi_A^{SC}, \Pi_B^{SC} - K$	$\Pi_A^{SC}, \Pi_B^{SC} - K$

(12)

In this matrix, firm B has a weakly dominant strategy to wait. Firm A will choose C against firm B's waiting if

$$\Pi_A^{CS} - K \geq \Pi_A^{SC},$$

or, equivalently,

$$K \leq \Pi_A^{CS} - \Pi_A^{SC} = \frac{2}{9}.$$

In this case (C,W) is the unique Nash equilibrium in undominated strategies, with outcome CS. Note that firm B is worse off compared to the simultaneous-move game.⁹ For the other case in which $K > 2/9$, (S,W) and (W,W) are Nash equilibria in undominated strategies. Both yield outcome SC, the same as in the simultaneous-move game. It is shown in the Appendix that both of the above two subregions of region IV are non-empty.¹⁰

5.6 Summary and Discussion

Let us summarize our analysis of the four regions. For the firm that does not have a dominant strategy in the simultaneous-move game, waiting is a weakly dominant strategy in the endogenous-timing game. Hence, the firm that has a dominant strategy in the simultaneous-move game (firm B in regions I and II, firm A in regions III and IV) can potentially enhance its position in the endogenous-timing game. Indeed, this firm can secure its equilibrium payoff from the simultaneous-move game by playing its dominant strategy in period 1 of the endogenous-timing game. However, it

⁹This is because $\Pi_A^{CS} - K \geq \Pi_A^{SC}$ and $\Pi_A^{SC} + \Pi_B^{SC} > \Pi_A^{CS} + \Pi_B^{CS}$ (Lemma 7) imply $\Pi_B^{CS} - K < \Pi_B^{SC}$.

¹⁰Matrix (12) has two Nash equilibria in weakly dominated strategies (S,C) and (W,C), with outcome SC.

can change its payoff by playing its dominated strategy in period 1. It will do so if that is beneficial. Specifically, we have the following.

- The firm that has a dominant strategy in the simultaneous-move game does not benefit from the endogenous-timing game in regions II and III and the upper portion of IV. Both firms receive the same payoffs in the endogenous-timing game as in the simultaneous-move game.
- The firm that has a dominant strategy in the simultaneous-move game benefits from the endogenous-timing game in region I and the lower portion of IV. The other firm is also better off in region I, but is worse off in the lower portion of region IV.

The next proposition points out the differences between equilibrium outcomes in the simultaneous-move and endogenous-timing games.

Proposition 2 (Endogenous-timing game versus simultaneous-move game). *The following comparison holds between the simultaneous-move and endogenous-timing games.*

- (i) *If $0.63 < q \leq 2.74$ and $K \in (\max\{0, c_1\}, c_2]$, then the endogenous-timing game results in both firms selecting standard products, whereas in the simultaneous-move game both firms customize.*
- (ii) *If $q > 2.84$ and $K \in (r_2, \min\{r_1, 2/9\}]$, then the endogenous-timing game results in firm A selecting a customized product and firm B selecting a standard product, whereas in the simultaneous-move game the opposite occurs.*

For all other parameter configurations the two games yield the same outcome.

Combining the results in this proposition with Figure 3 leads to Figure 5, which presents the equilibrium customization choices in the endogenous-timing game.

It follows from Propositions 1 and 2 that both quality difference and endogenous timing play important roles in determining the equilibrium outcome. Indeed, if the quality difference is small ($q \leq 0.56$) then no firm customizes in both simultaneous-move and endogenous-timing games, regardless of the level of K . However, when the quality difference is large, firms are sufficiently differentiated. Customization by one or both firms may occur and the simultaneous-move and endogenous-timing games do not always lead to the same outcome. In the endogenous-timing game, the firm that does not have a dominant strategy in the simultaneous-move game has an incentive to wait, and the firm that has a dominant strategy can achieve a different outcome from that of the simultaneous-move game by playing its dominated strategy in period 1.

6 Conclusion

In this paper we have studied customization in the presence of quality differentiation. Our model has two firms that are initially differentiated horizontally and vertically. Customization affects horizontal but not vertical differentiation. We show that product quality affects the firms' equilibrium

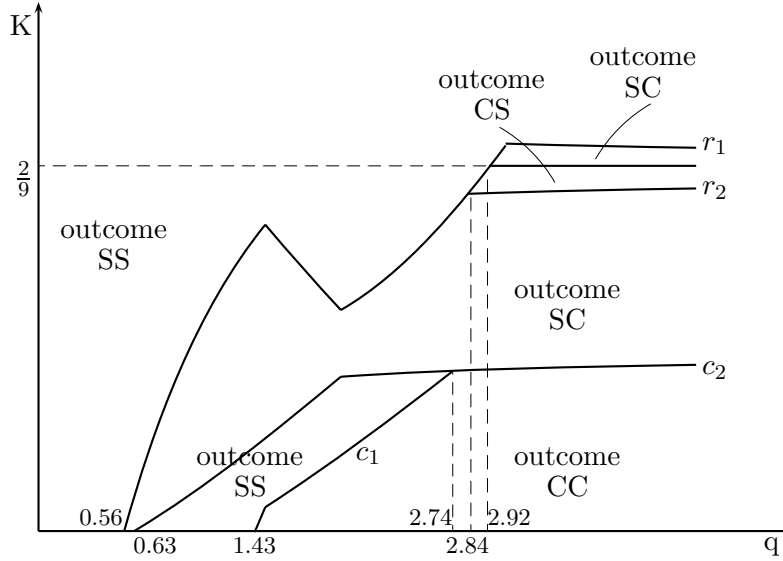


Figure 5: Equilibrium customization choices in the endogenous-timing game

customization choices. In particular, no firm will customize if the quality difference is small, regardless of the fixed cost of customization. For sufficiently large quality differences, customization by one or both firms may occur. Because the high quality firm benefits more from customization than the low quality firm, the high quality firm is more likely to customize. This result is supported by many real-world observations. We do not see customized low quality bicycles and shoes. As another example, Timbuk2 customizes its messenger bags, whereas lower quality bags made by many other manufactures are not customized.

Endogenous timing in customization choices may enable the firms to achieve a Pareto superior outcome by avoiding a price war that would follow customization by both firms. This happens when the high quality firm is the only firm with a dominant strategy in the simultaneous-move game. While in the simultaneous-move game the low quality firm never customizes alone, in the endogenous-timing game it can obtain an advantage by becoming the first and only firm to customize. This happens when the low quality firm is the only firm with a dominant strategy in the simultaneous-move game.

There are a number of directions that can be taken to extend the present paper. One extension is to make the range of customization a continuous choice variable. Such a model would involve a fairly complex analysis. We believe that the qualitative results of our paper will hold; e.g., the range of customization of the higher quality firm is larger than that of the lower quality firm. Another interesting extension is to introduce dynamics into the model to address the roles of customization and quality in entry deterrence.

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Appendix

Proof of Lemma 1. Each case is proven in turn.

- (i) Consider $q \leq 3/2$ and suppose the indifference line (5) intersects the unit square as shown in Figure 2(a). Straightforward algebra implies

$$D_A(p_A, p_B) = \frac{1}{2} \left(1 - \frac{1}{2}q + p_B - p_A \right) \quad \text{and} \quad D_B(p_A, p_B) = \frac{1}{2} \left(1 + \frac{1}{2}q + p_A - p_B \right)$$

in this case. The profit maximizing first-order conditions yield the equilibrium prices and profits as in part (i) of the lemma. It is left to verify that under these prices the indifference line intersects the top and bottom sides of the unit square. Algebraically, $\hat{x}(1) \geq 0$ and $\hat{x}(0) \leq 1$. Indeed,

$$\hat{x}(1) = \frac{1}{2} (1 - q + p_B^{SS} - p_A^{SS}) = \frac{1}{2} \left(1 - \frac{2}{3}q \right) \geq 0,$$

$$\hat{x}(0) = \frac{1}{2} (1 + p_B^{SS} - p_A^{SS}) = \frac{1}{2} \left(1 + \frac{1}{3}q \right) < 1$$

hold for $q \leq 3/2$. In fact, $\hat{x}(1) = 0$ when $q = 3/2$.

- (ii) Consider $3/2 < q \leq 3$ and suppose the indifference line (5) intersects the unit square as shown in Figure 2(b). The firms' demand functions are

$$D_A(p_A, p_B) = \frac{1}{4q} (1 + p_B - p_A)^2 \quad \text{and} \quad D_B(p_A, p_B) = 1 - \frac{1}{4q} (1 + p_B - p_A)^2.$$

The first-order conditions yield the equilibrium prices and profits as in part (ii) of the lemma. Under these prices the indifference line intersects the left and bottom sides of the unit square. Algebraically, $\hat{x}(1) \leq 0$ and $\hat{x}(0) \in [0, 1]$. Indeed,

$$\hat{x}(1) = \frac{1}{2} \left(1 - q + \frac{1}{4} \left(-3 + \sqrt{1 + 16q} \right) \right) < 0,$$

$$\hat{x}(0) = \frac{1}{2} \left(1 + \frac{1}{4} \left(-3 + \sqrt{1 + 16q} \right) \right) \in (0, 1]$$

hold for $3/2 < q \leq 3$. Note that $\hat{x}(1) = 0$ when $q = 3/2$ and $\hat{x}(0) = 1$ when $q = 3$.

- (iii) Consider $q > 3$ and suppose the indifference line (5) intersects the unit square as shown in Figure 2(c). The firms' demand functions are

$$D_A(p_A, p_B) = \frac{1}{q} (p_B - p_A) \quad \text{and} \quad D_B(p_A, p_B) = \frac{1}{q} (q + p_A - p_B).$$

The first-order conditions yield the equilibrium prices and profits as in part (iii) of the lemma. Under these prices the indifference line intersects the right and left sides of the unit square.

Algebraically, $\hat{x}(1) \leq 0$ and $\hat{x}(0) \geq 1$. Indeed,

$$\hat{x}(1) = \frac{1}{2} \left(1 - \frac{2}{3}q \right) < 0 \quad \text{and} \quad \hat{x}(0) = \frac{1}{2} \left(1 + \frac{1}{3}q \right) > 1$$

hold for $q > 3$. Note that $\hat{x}(0) = 1$ when $q = 3$.

□

Proof of Lemma 2. Each case is proven in turn.

(i) Consider $q \leq 1$ and suppose the indifference line (6) intersects the unit square as shown in Figure 2(a). The firms' demand functions are

$$D_A(p_A, p_B) = 1 - \frac{1}{2}q + p_B - p_A \quad \text{and} \quad D_B(p_A, p_B) = \frac{1}{2}q + p_A - p_B.$$

The first-order conditions yield the equilibrium prices and profits as in part (i) of the lemma. Under these prices the indifference line intersects the top and bottom sides of the unit square. Algebraically, $\hat{x}(1) \geq 0$ and $\hat{x}(0) \leq 1$. Indeed,

$$\hat{x}(1) = 1 - q + p_B^{CS} - p_A^{CS} = \frac{2}{3} - \frac{2}{3}q \geq 0 \quad \text{and} \quad \hat{x}(0) = 1 + p_B^{CS} - p_A^{CS} = \frac{2}{3} + \frac{1}{3}q < 1$$

hold for $q \leq 1$. Note that $\hat{x}(1) = 0$ and $\hat{x}(0) = 1$ when $q = 1$.

(ii) Consider $q > 1$ and suppose the indifference line (6) intersects the unit square as shown in Figure 2(c). The firms' demand functions are

$$D_A(p_A, p_B) = \frac{1}{q} \left(\frac{1}{2} + p_B - p_A \right) \quad \text{and} \quad D_B(p_A, p_B) = \frac{1}{q} \left(q - \frac{1}{2} + p_A - p_B \right).$$

The first-order conditions yield the equilibrium prices and profits as in part (ii) of the lemma. Under these prices the indifference line intersects the right and left sides of the unit square. Algebraically, $\hat{x}(1) \leq 0$ and $\hat{x}(0) \geq 1$. Indeed,

$$\hat{x}(1) = \frac{2}{3} - \frac{2}{3}q < 0 \quad \text{and} \quad \hat{x}(0) = \frac{2}{3} + \frac{1}{3}q > 1$$

hold for $q > 1$. Note that $\hat{x}(1) = 0$ and $\hat{x}(0) = 1$ when $q = 1$.

□

Proof of Lemma 3. Each case is proven in turn.

(i) Consider $q \leq 1/2$ and suppose the indifference line (7) intersects the unit square as shown in Figure 2(a). The firms' demand functions are

$$D_A(p_A, p_B) = -\frac{1}{2}q + p_B - p_A \quad \text{and} \quad D_B(p_A, p_B) = 1 + \frac{1}{2}q + p_A - p_B.$$

The first-order conditions yield the equilibrium prices and profits as in part (i) of the lemma. Under these prices the indifference line intersects the top and bottom sides of the unit square. Algebraically, $\hat{x}(1) \geq 0$ and $\hat{x}(0) \leq 1$. Indeed,

$$\hat{x}(1) = -q + p_B^{SC} - p_A^{SC} = \frac{1}{3} - \frac{2}{3}q \geq 0 \quad \text{and} \quad \hat{x}(0) = p_B^{SC} - p_A^{SC} = \frac{1}{3} + \frac{1}{3}q < 1$$

hold for $q \leq 1/2$. Note that $\hat{x}(1) = 0$ when $q = 1/2$.

(ii) Consider $1/2 < q \leq 2$ and suppose the indifference line (7) intersects the unit square as shown in Figure 2(b). The firms' demand functions are

$$D_A(p_A, p_B) = \frac{1}{2q} (p_B - p_A)^2 \quad \text{and} \quad D_B(p_A, p_B) = 1 - \frac{1}{2q} (p_B - p_A)^2.$$

The first-order conditions yield the equilibrium prices and profits as in part (ii) of the lemma. Under these prices the indifference line intersects the left and bottom sides of the unit square. Algebraically, $\hat{x}(1) \leq 0$ and $\hat{x}(0) \in [0, 1]$. Indeed,

$$\hat{x}(1) = -q + \frac{1}{2}\sqrt{2q} < 0 \quad \text{and} \quad \hat{x}(0) = \frac{1}{2}\sqrt{2q} \in (0, 1]$$

hold for $1/2 < q \leq 2$. Note that $\hat{x}(1) = 0$ when $q = 1/2$ and $\hat{x}(0) = 1$ when $q = 2$.

(iii) Consider $q > 2$ and suppose the indifference line (7) intersects the unit square as shown in Figure 2(c). The firms' demand functions are

$$D_A(p_A, p_B) = \frac{1}{q} \left(-\frac{1}{2} + p_B - p_A \right) \quad \text{and} \quad D_B(p_A, p_B) = \frac{1}{q} \left(q + \frac{1}{2} + p_A - p_B \right).$$

The first-order conditions yield the equilibrium prices and profits as in part (iii) of the lemma. Under these prices the indifference line intersects the right and left sides of the unit square. Algebraically, $\hat{x}(1) \leq 0$ and $\hat{x}(0) \geq 1$. Indeed,

$$\hat{x}(1) = \frac{1}{3} - \frac{2}{3}q < 0 \quad \text{and} \quad \hat{x}(0) = \frac{1}{3} + \frac{1}{3}q > 1$$

hold for $q > 2$. Note that $\hat{x}(0) = 1$ when $q = 2$.

□

Proof of Lemma 4. The results follow immediately from the first-order conditions. □

Proof of Lemma 5. The expressions for c_1 , c_2 , r_1 , and r_2 as functions of q follow immediately from

Lemmas 1 through 4,

$$c_1 = \begin{cases} \left(\frac{2}{3} - \frac{1}{6}q\right)^2 - \frac{1}{2}\left(1 - \frac{1}{6}q\right)^2, & \text{if } q \leq 1 \\ \frac{1}{q}\left(\frac{1}{3}q + \frac{1}{6}\right)^2 - \frac{1}{2}\left(1 - \frac{1}{6}q\right)^2, & \text{if } q \in \left(1, \frac{3}{2}\right] \\ \frac{1}{q}\left(\frac{1}{3}q + \frac{1}{6}\right)^2 - \frac{1}{q}\left(\frac{1+\sqrt{1+16q}}{8}\right)^3, & \text{if } q \in \left(\frac{3}{2}, 3\right] \\ \frac{1}{q}\left(\frac{1}{3}q + \frac{1}{6}\right)^2 - \frac{1}{9}q = \frac{1}{9} + \frac{1}{36q}, & \text{if } q > 3 \end{cases}$$

$$c_2 = \begin{cases} \frac{1}{9}q - \left(\frac{1}{3} - \frac{1}{6}q\right)^2, & \text{if } q \leq \frac{1}{2} \\ \frac{1}{9}q - \frac{1}{16}\sqrt{2q}, & \text{if } q \in \left(\frac{1}{2}, 2\right] \\ \frac{1}{9}q - \frac{1}{q}\left(\frac{1}{3}q - \frac{1}{6}\right)^2 = \frac{1}{9} - \frac{1}{36q}, & \text{if } q > 2 \end{cases}$$

$$r_1 = \begin{cases} \left(\frac{2}{3} + \frac{1}{6}q\right)^2 - \frac{1}{2}\left(1 + \frac{1}{6}q\right)^2, & \text{if } q \leq \frac{1}{2} \\ \frac{9}{16}\sqrt{2q} - \frac{1}{2}\left(1 + \frac{1}{6}q\right)^2, & \text{if } q \in \left(\frac{1}{2}, \frac{3}{2}\right] \\ \frac{9}{16}\sqrt{2q} - \left(1 - \frac{1}{q}\left(\frac{1+\sqrt{1+16q}}{8}\right)^2\right)\frac{-5+3\sqrt{1+16q}}{8}, & \text{if } q \in \left(\frac{3}{2}, 2\right] \\ \frac{1}{q}\left(\frac{2}{3}q + \frac{1}{6}\right)^2 - \left(1 - \frac{1}{q}\left(\frac{1+\sqrt{1+16q}}{8}\right)^2\right)\frac{-5+3\sqrt{1+16q}}{8}, & \text{if } q \in (2, 3] \\ \frac{1}{q}\left(\frac{2}{3}q + \frac{1}{6}\right)^2 - \frac{4}{9}q = \frac{2}{9} + \frac{1}{36q}, & \text{if } q > 3 \end{cases}$$

and

$$r_2 = \begin{cases} \frac{4}{9}q - \left(\frac{1}{3} + \frac{1}{6}q\right)^2, & \text{if } q \leq 1 \\ \frac{4}{9}q - \frac{1}{q}\left(\frac{2}{3}q - \frac{1}{6}\right)^2 = \frac{2}{9} - \frac{1}{36q}, & \text{if } q > 1 \end{cases}$$

Tedious but straightforward numerical calculations confirm that $c_i < r_j$ for $i, j = 1, 2$ and any given value of q . \square

Proof of Proposition 1. The results follow immediately from Lemma 5 and the discussion preceding Proposition 1. \square

Proof of Lemma 6. The expressions for Π_A^{SS} and Π_B^{SS} as functions of q are given in Lemma 1. Numerical calculations confirm that $\Pi_A^{SS} > \Pi_A^{CC} = q/9$ and $\Pi_B^{SS} > \Pi_B^{CC} = 4q/9$ for $q < 3$. For $q \geq 3$, $\Pi_A^{SS} = \Pi_A^{CC}$ and $\Pi_B^{SS} = \Pi_B^{CC}$. \square

Proof of Lemma 7. It follows from Lemmas 2 and 3 that

$$\Pi_A^{CS} + \Pi_B^{CS} = \begin{cases} \frac{5}{9} - \frac{1}{9}q + \frac{1}{18}q^2, & \text{if } q \leq 1 \\ \frac{1}{q}\left(\frac{5}{9}q^2 - \frac{1}{9}q + \frac{1}{18}\right), & \text{if } q > 1 \end{cases}$$

and

$$\Pi_A^{SC} + \Pi_B^{SC} = \begin{cases} \frac{5}{9} + \frac{1}{9}q + \frac{1}{18}q^2, & \text{if } q \leq 1/2 \\ \frac{5}{8}\sqrt{2q}, & \text{if } q \in \left(\frac{1}{2}, 2\right] \\ \frac{1}{q}\left(\frac{5}{9}q^2 + \frac{1}{9}q + \frac{1}{18}\right), & \text{if } q > 2 \end{cases}$$

Obviously

$$\Pi_A^{CS} + \Pi_B^{CS} > \Pi_A^{SC} + \Pi_B^{SC}$$

holds for $q \leq 1/2$ and $q > 2$. Straightforward calculations show that the inequality also holds for $q \in (1/2, 2]$. \square

Proof of (10). It suffices to show that

$$\Pi_B^{SC} - \Pi_B^{CS} > c_1$$

holds for $q > 2.74$. Lemmas 2 and 3 imply $\Pi_B^{SC} - \Pi_B^{CS} = 4/9$. Straightforward calculations confirm that $4/9 > c_1$ for $q > 2.74$. \square

Proof of non-emptiness of the two subregions of region IV. For $q > 3$,

$$r_1 = \frac{2}{9} + \frac{1}{36q} \quad \text{and} \quad r_2 = \frac{2}{9} - \frac{1}{36q}.$$

Hence,

$$r_2 < \frac{2}{9} < r_1,$$

from which non-emptiness of the two subregions of region IV follows. \square

Proof of Proposition 2. The results follow immediately from Proposition 1 and the discussion of the four regions preceding this proposition. \square