Unconventional Optimal Repurchase Agreements*

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Abstract

We build a model in which verifiability of private debts, timing mismatch in debt settlements and borrowing leverage lead to liquidity crisis in the financial market. Central bank can respond to the liquidity crisis by adopting an unconventional monetary policy that resembles repurchase agreements between the central bank and the lenders. This policy is effective if the timing mismatch is nominal (i.e., a settlement participation risk). It is ineffective if the timing mismatch is driven by a real shock (i.e., preference shock).

Keywords: liquidity problem, timing mismatch, leveraging, liquidity shock, settlement risk, repurchase agreement, consumption shock

JEL classification: E44, E52, G01

1 Introduction

Financial crises are rare events in the United States. Yet, in those rare events, liquidity problems attract a great deal of attention. Indeed, at the onset of the 2007 Financial Crisis, the Federal Reserve adopted radical new facilities in order to address the liquidity problems. From January 2008 to January 2010, the Fed's monetary liabilities increased from $800 billion to $2 trillion. Despite a large supply of treasury securities available, the Fed began directly purchasing private credit market instruments.[1] Why did the Federal Reserve resort to directly buying private credit

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[1]Schreft and Smith (2002) analyzed the conduct of monetary policy in a setting in which the quantity of government debt was small and shrinking. Private securities may have to be used in open market operations when no treasury debt is outstanding.

instruments in order to expand liquidity? More importantly, under what conditions are these new facilities welfare improving?

To study these questions, we build a model in a three-period overlapping generations setup in which the two basic elements–fiat money and private debts–coexist. Fiat money is valued because it facilitates intergenerational transactions. Within a generation, some agents are endowed with capital (they will become lenders in our model) and the others have access to short-term and long-term production technologies (these are the borrowers). Borrowers issue debts to acquire capital to produce. Debt contracts, or IOUs, can be settled either one period (short-term) or two periods (long-term) after issue. Long-term IOUs bear higher interest as the return to long-term production is higher. IOUs can be settled by two types of assets–consumption good, which is produced by the production technologies and is valued by the lenders, and fiat money, which can be used to purchase the consumption good.[2]

To be clear, we do not attempt to explicitly model the 2007-2009 financial crisis. Instead, we build an environment with typical frictions in the financial market in which unconventional monetary policy can play a role. In our approach, three typical frictions–verifiability of private debts, timing mismatches in private debt settlement and leverage–are essential to a liquidity problem. Private debt contracts are not verifiable everywhere, and cannot circulate as a means of payments for consumption goods even in absence of default risk. Therefore, private debt has to be redeemed at the settlement time. However, the settlement between lenders and borrowers is subject to timing mismatch. Lenders face uncertainty in whether they can participate in long-term debt settlements. If they cannot, they sell long-term debts in a secondary market at a price determined by the available liquidity.

The verifiability problem and timing mismatch provide trading opportunities in the secondary market. But it is the leverage that drives the liquidity shortage in the secondary market, and thus the liquidity problem.[3] In the short term, borrower's available liquidity is accumulated fiat money

[2]The settlement process described here is essentially the same as that put forward in Freeman (1996). We modify it so that units of the consumption good are also acceptable as a settlement device. We can say that private debt in this model economy is consistent with the real bills doctrine.

[3]Lippman and McCall (1986) provide an excellent overview of the concept of liquidity. Throughout our analysis, we adopt the notion put forward by Hirshleifer (1968), who argued that liquidity is "an asset's capability over time of being realized in the form of funds available for immediate consumption or reinvestment–proximately in the form of money" (p.1). Outside money can be used generally to obtain consumption goods because it is verifiable while inside money is more limited because it is not verifiable.

balances. If long-term borrowers are able to accumulate sufficient money balances to settle debts before their investment matures, debts will be settled at par. Otherwise, debts will be sold at a discount price. That is, the liquidity problem arises only if long-term borrowers borrow more than what they can settle with money in the short term, or in other words, only if they are leveraged when they borrow.[4]

The liquidity problem causes misallocation of consumption and production. The lenders who are excluded from long-term debt settlements see their consumption decline relative to those who are not subject to the participation retrictrion. The reason is simple, restricted lenders sell their IOUs at a discount. When debt sells at a discount price, there is an opportunity to make profit in the seondary market, inducing some borrowers to choose the lower return, short-term technology.

The central bank can address the liquidity problem using a three-step policy that is observationally equivalent to a repurchase agreement. The central bank purchases private debt contracts at par in the secondary market using money, holds the IOUs for one period and then settles with the issuers. Finally, the bank accepts fiat money in the consumption good market by selling the goods that are received from settlement, taking the money out of circulation. We refer to this policy as an unconventional repurchase agreement. Unconventional in the sense that the central bank is purchasing a private credit instrument in the open market.[5] Because of the overlapping generations structure of our model, the central bank operates purchase and repurchase simultaneously (but to different generations) in a period, and the aggregate money stock is constant over time. Hence, the unconventional policy redistributes liquidity to different markets. The liquidity injected in the secondary market raises the price of debt to par and eliminates profits in the secondary market. Thus, all capital is invested in the higher return, long-term production and efficiency is achieved.

The central bank is able to solve the liquidity problem because it has two advantages over individual agents. First, it can costlessly produce fiat money and replace the unverifiable private debts with verifiable fiat money. Second, the bank is not subject to the settlement participation restrictions. So the bank literally holds the private IOUs for the lenders and returns the matured IOUs to the lenders after one period. We consider these two advantages to be the basic elements of the functions that define a central bank. In an economy with typical financial frictions present,

[4] See Holmstrom and Tirole (2011) for a description of the interaction between liquidity and leveraging.

[5] Hence, the liquidity problem is solved by a modified version of Gertler and Karadi's (2011) unconventional monetary policy.

unconventional monetary policy is useful simply due to the central bank's fundamental functions.

The literature discusses unconventional monetary policy from an agency different perspective. Gertler and Karadi (2011) investigate unconventional monetary policy in a model in which principal-agent problems exist between the lenders and borrowers, but not between the central bank and lenders. The central bank can act like an intermediary during a financial crisis, eliminating the principal-agent problem and stimulating output. From a mechanism design standpoint, the question is whether the commercial banks would be sustainable in the long run or whether the central bank would emerge as the sole financial intermediary.[6]

We extend the baseline model economy in two ways. First, we consider a case in which there are generation-specific goods. As opposed to a single consumption good, there is a good that is preferred by each generation. We demonstrate that this modification has no effect on the optimal unconventional monetary policy. Second, we modify the lender's preference. Instead of having a settlement participation risk, the lenders are subject to preference shocks. Some lenders have to settle before long-term production completes because they only value middle-aged, or early, consumption. With such a change to preferences, the secondary market price is efficient even though the price is below the par in some cases. Here the discount price measures the relative cost of consumption. Early consumption is more costly as it requires some resources to be allocated in the lower return, short-term technology. Because the laissez-faire price is efficient, policy intervention is unnecessary and can be harmful. A lesson we learn from this exercise is that when the secondary market price falls below par, it does does not necessarily justify policy intervention. Identifying the driving force behind the discount price–whether it is due to a pure settlement restriction or a preference shock–is important.

In a closely related paper, Williamson (2011) studies a broad range of monetary and fiscal policy actions in a New Monetarist model economy. In presence of a real shock to production, the central bank's lending to producers is redundant, unless it offers better terms to producers than they would receive in the private sector, in which case central bank's lending only has redistributional effect. Our result confirms Williamson's view on unconventional monetary policy; it is ineffective if the underlying shock is a real one.

[6] There is extensive literature studying principal-agent problem that results in financial crisis and amplifies shocks at business cycles frequencies. See Kiyotaki and Moore (2008). See also Hall (2011) for and excellent, brief overview of the literature on principal-agent problems and economic activity.

The outline of the paper is as follows. Section 2 lays out the physical environment. Section 3 solves for the equilibrium in the baseline model. Section 4 solves the planner's allocation. Section 5 describes the repurchase agreement that implements the planner's allocation. Sections 6 and 7 discuss two extensions to the baseline model. Section 8 concludes.

2 Environment

There is an infinite sequence of time periods. Dates are indexed by $t = 1, 2, 3, \ldots$. There are two types of agents: type 1 and type 2. Each period, a continuum of measure one of each type are born. Both types of agents live for three periods. At date $t = 1$, there is the initial old generation that lives for one period and the initial middle-aged generation that lives for two periods. For both the intial-old generation and the initial middle-aged generation, there is a continuum of measure one of each type.

The commodity space consists of single, perishable consumption goods and physical capital. Each type 1 is endowed with κ units of the capital good when young and nothing middle-aged or old. Meanwhile, each type 2 agent is endowed with one unit of labor when young. We assume that at each date labor is transformed into units of the consumption good at a one-for-one rate. In addition, young type 2 agents have access to two production technologies that turn capital into future units of the consumption good. The short-term technology and long-term technology are represented by production functions $f(k)$ and $Af(k)$, respectively. We assume that $A > 1$ and that $f(k)$ is continuously differentiable, strictly increasing and strictly concave. For any date-t investment, the short-term technology yields output at date $t+1$, whereas long-term technology yields output at date $t+2$. Without loss of generality, we assume that the a type 2 agent is engaged in either the short-term or the long-term technology. Let λ be the measure of type 2 agents who choose the short-term technology (we refer to these type 2 agents as short-term producers) and $1-\lambda$ be the measure of type 2 who choose the long-term technology (we call them long-term producers). Capital depreciates fully after production completes. The initial old and initial middle-aged type 1 agents own a fixed stock of fiat money, totaling M_0 dollars. The initial old and the initial middle-aged type 2 agents are endowed with consumption good.

Type 1 agents value the consumption good when they are middle-aged and old. Let $v(q_{2t+1} + q_{3t+2})$

denote a type 1's preference, where q is type 1's consumption. In general, when there are two subscripts, the first denotes period of life and the second denotes the date. Type 2 agents value consumption when old and derive disutility from working when young. Let $-g(l_{1t}) + u(x_{3t+2})$ represent a type 2's preference, where l is the labor used when young and x is the consumption when old. We assume that $v(\cdot)$ and $u(\cdot)$ are strictly increasing, strictly concave and differentiable, $g(\cdot)$ is strictly increasing, strictly convex, differentiable, and satisfies $\lim_{l \to 0} g'(l) = 0$ and $\lim_{l \to 1} g'(l) = \infty$.

At each date, settlement and markets open sequentially. In order, the capital market trades, settlement meetings occur, private IOUs trade in the secondary market, and the consumption good market trades. Type 2 agents participate in all exchanges during their life. A type 1 agent participates in all exchange during his life with probability α and with probability $1-\alpha$ he is excluded from settlement when old. Type 1 agents learn their market participation pattern when middle-aged. By law of large numbers, a fraction of $1-\alpha$ type 1 agents are excluded from settlement when old.

In the capital market, young type 2 agents acquire capital from young type 1 agents. The young type 2 agents then decide to employ the capital in either the short-term or the long-term technology. Because young type 2 agents have neither money nor consumption goods, they issue private debt contracts to the young type 1 agents in exchange for the capital. Accordingly, we refer to type 1 agents as lenders and type 2 agents as borrowers. The debt contracts charge different interest rates depending on when the IOUs are settled. Without loss of generality, we assume the nominal interest rate on IOU is 1 for those settled when borrowers are middle-aged. If settled when old, the borrower pays the nominal (gross) interest rate $1+\gamma$. IOUs can be settled by either fiat money or the consumption good. During settlement, debt contracts are costlessly enforced; that is, there is no default risk. However, private IOUs are not verifiable in the consumption good market. Because of the verifiability concern, debt contracts cannot be used in future transactions as a means of payment.

The temporal mismatch becomes evident at settlement. Note that both middle-aged and old borrowers are present at settlement, but some old lenders are excluded. For borrowers choosing the short-term technology, they settle IOUs when middle-aged. For long-term producers, they may not be able to settle when middle-aged. Long-term producers can use their accumulated money balances to settle some of IOUs. Middle-aged lenders can sell unsettled IOUs in the secondary market. We call the lenders who sell (buy) IOUs in the secondary market when middle-aged as early-settling (late-settling) lenders. Denote the measure of early-settling lenders by $1 - \tilde{\alpha}$ with $\max\{\tilde{\alpha}\} = \alpha$. That is, the lenders who are excluded from settlement when old have to sell unsettled IOUs in the secondary market. Producers who have settled all their debt contracts can buy IOUs in the secondary market.

Lastly, the consumption goods market opens. The sellers are young borrowers who produce consumption goods using labor. The buyers are the middle-aged and the old borrowers and lenders. In consumption good market, buyers trade fiat money for consumption good. This explains how borrowers have fiat money in debt settlement. Note that borrowers cannot participate in the secondary market or goods market before they settle all IOUs.

3 The baseline model

3.1 Lenders

The lender's problem is straightforward to solve and thus where we start. With the lender's preferences, capital is inelastically lent to the borrowers. Because we assume the technology decision is private information, the lender sells the capital to short-term and long-term producers at the same price. The nominal value of the debt contract is $p_{kt}\kappa$, where p_{kt} is the price of capital at time t. For early-settling lenders, unsettled IOUs are sold in the secondary market when middle-aged. The early-settling lender then buys consumption goods from the young borrowers in the goods market. Therefore, the early-settling lenders date-t budget constraint is

$$\left[\rho_{t+1}\left(1+\gamma_t\right)\left(1-a_t\right)+a_t\right] p_{kt}\kappa = \min\left\{p_{xt+1}, p_{xt+2}\right\}\left(q_{2t+1}+q_{3t+2}\right) \tag{1}$$

where p_{xt} is the price of consumption goods at time t, γ_t is the net interest rate on the IOUs issued at t and settled at $t+2$, a_t is the fraction of the debt contracts that are settled when middle-aged, and ρ_{t+1} is the price of IOUs in the secondary market at time $t+1$. The value of a_t is determined by the borrower's production choice, which will be clear at the end of the section. The early-settling lenders have the choice of consuming when middle-aged or old, depending on the period in which goods are cheaper.

A late-settling lender purchases IOUs in the secondary market, receiving settlement next period. A late-settling lender uses money and goods obtained from settled IOUs, i.e., $a_t p_{kt} \kappa$, to purchase unsettled debt contracts. A late-settling lender's budget constraint is

$$\frac{\rho_{t+1}(1+\gamma_t)(1-a_t)+a_t}{\rho_{t+1}} p_{kt}\kappa = p_{xt+2} q^*_{3t+2} \tag{2}$$

In what follows, we use superscript "*" to denote the variables associated with late-settling lenders and long-term producers.

Lemma 1 *In equilibrium, the interest rate satisfies* $\rho_{t+1}(1+\gamma_t) = 1$, *and* $\rho_{t+1} = \gamma_t = 1$ *if* $a_t = 1$.

By Lemma 1, the lender's consumption is determined by the budget constraints. Formally,

$$q_{2t+1} + q_{3t+2} = \frac{p_{kt}\kappa}{\min\{p_{xt+1}, p_{xt+2}\}} \tag{3}$$

$$q^*_{3t+2} = \frac{1}{\rho_{t+1}} \frac{p_{kt}\kappa}{p_{xt+2}}, \; q^*_{2t+1} = 0 \tag{4}$$

Note that early-settling lenders' total income is equal to the nominal value of the debt, whereas late-settling lenders earn extra interest of γ_t on the IOUs minus the inflation rate.

3.2 Short-term producers

When young, a short-term producer borrows from young lenders and uses labor to produce for the middle-aged and the old agents. Let h_t be the nominal value of IOUs that the short-term producer issues at date t. Capital purchases are linked to the size of the debt by the following expression

$$h_t = p_{kt} k_t \tag{5}$$

With his labor, the young borrower produces consumption good, trading it for fiat money. At the end of date t, the short-term producer's money balances, denoted by m_t, are

$$m_t = p_{xt} l_{1t} \tag{6}$$

When middle-aged, the short-term producer completes production. After debt settlement, the short-term producer has nominal resources worth $p_{xt+1} f(k_t) + m_t - h_t$, which he uses to purchase

IOUs in the secondary market.[7] When these IOUs are finally settled in the next period, the short-term producer gets $(1+\gamma_t)\left[p_{xt+1}f(k_t)+m_t-h_t\right]$, which finances his old-age consumption. From Lemma 1, the old short-term producer's budget constraint when he is old is

$$\frac{p_{xt+1}f(k_t)+m_t-h_t}{\rho_{t+1}}=p_{xt+2}x_{3t+2} \tag{7}$$

Plug $(5)-(6)$ to (7) to get the short-term producer's life-time budget constraint:

$$p_{xt+1}f(k_t)+p_{xt}l_{1t}-p_{kt}k_t=\rho_{t+1}p_{xt+2}x_{3t+2} \tag{8}$$

A short-term producer maximizes lifetime utility subject to (8). The first-order conditions are

$$f'(k_t)-\frac{p_{kt}}{p_{xt+1}} = 0 \tag{9}$$

$$g'(l_{1t})-\frac{p_{xt}}{\rho_{t+1}p_{xt+2}}u'(x_{3t+2}) = 0 \tag{10}$$

3.3 Long-term producers

A long-term producers solves a similar problem as a short-term producer. Let h_t^* and m_t^* be the nominal value of the IOUs and money balances that a young long-term producer has. When middle aged, the long-term producer settles some, if not all, IOUs with money. If money is sufficient to cover the debt, i.e., $m_t^*\geq h_t^*$, the long-term producer can apply any remaining liquidity to purchase IOUs in the secondary market with gross return of $1/\rho_{t+1}$.[8] In contrast, with $m_t^*<h_t^*$ the long-term producer must redeem remaining indebtedness when old, paying $1+\gamma_t$ after long-term production is complete. Long-term producers finance their old-age consumption using after-debt income. By Lemma 1, the long-term producer's old-age budget constraint can be rewritten as

$$p_{xt+1}Af(k_t^*)+\frac{1}{\rho_{t+1}}\left(m_t^*-h_t^*\right)=p_{xt+2}x_{3t+2}^* \tag{11}$$

Plug in $m_t^*=p_{xt}l_{1t}^*$ and $h_t^*=p_{kt}k_t^*$ to get the long-term producer's lifetime budget constraint as

[7] Note that the short-term producers do not have to participate in the secondary market. Instead, they can participate in the goods markets and arbitrage between different time periods if prices are not stationary. In equilibrium, the short-term producers must be indifferent to these two options. Here we assume they participate in the secondary market only, as later we will focus on the stationary equilibrium where arbitraging between time periods result in zero profit.

[8] Note that the long-term producers can choose not to participate in the secondary market. See footnote 6. Here we consider the case that they participate in the secondary market, as later we will focus on the stationary equilibrium.

$$\rho_{t+1}p_{xt+1}Af(k_t^*) + p_{xt}l_{1t}^* - p_{kt}k_t^* = \rho_{t+1}p_{xt+2}x_{3t+2}^* \tag{12}$$

The first-order conditions are

$$Af'\left(k_t^*\right) - \frac{1}{\rho_{t+1}}\frac{p_{kt}}{p_{xt+1}} = 0 \tag{13}$$

$$g'\left(l_{1t}^*\right) - \frac{p_{xt}}{\rho_{t+1}p_{xt+2}}u'\left(x_{3t+2}^*\right) = 0 \tag{14}$$

A borrower chooses production technology by comparing the lifetime utility of being a short-term or a long-term producer. If the two technologies yield the same utility, he chooses the short-term technology with probability λ. By law of large numbers, the fraction of short-term producers is λ. Given the fraction of short-term producers, the fraction of debts that are settled when the borrowers are middle-aged is

$$a_t = \begin{cases} \dfrac{\lambda_t h_t + (1-\lambda_t)\, m_t^*}{p_{kt}\kappa} & \text{if } h_t^* > m_t^* \\ 1 & \text{otherwise.} \end{cases}$$

3.4 Market clearing

The competitive market clearing conditions are

$$\begin{aligned} \kappa &= \lambda_t k_t + (1-\lambda_t)\, k_t^*, \\ \lambda_t l_t + (1-\lambda_t)\, l_t^* + \lambda_{t-1} f\left(k_{t-1}\right) + (1-\lambda_{t-2})\, Af\left(k_{t-2}^*\right) &= \lambda_{t-2}x_{3t} + (1-\lambda_{t-2})\, x_{3t}^* + \tilde{\alpha}_{t-2}q_{3t}^* \\ &\quad + (1-\tilde{\alpha}_{t-2})\, q_{3t} + (1-\tilde{\alpha}_{t-1})\, q_{2t}. \end{aligned}$$

The first equation is the market clearing condition for capital. The second equation represents the market-clearing condition for the consumption good. Consumption goods are supplied by young borrowers, middle-aged short-term producers and old-aged long-term producers. Old borrowers, old lenders, and middle-aged early-settling lenders demand consumption goods.

The money market-clearing condition is

$$M_0 = \lambda_t m_t + (1-\lambda_t)\, m_t^*$$

The loan market-clearing condition is

$$p_{kt}\kappa = \lambda_t h_t + (1-\lambda_t)\, h_t^*$$

With Lemma 1 and the first-order conditions from the producer's problem, the secondary market-clearing condition can be written as

$$\lambda_{t-1}\left[f(k_{t-1})+l_{1t-1}\right]+(1-\lambda_{t-1})\,l_{1t-1}^{*}\geq(1-\tilde{\alpha}_{t-1})\,\rho_{t}Af'\left(k_{t-1}^{*}\right)\kappa \tag{15}$$

The left-hand side is the supply of liquidity in real terms. It comes from two sources–short-term producer's output and the borrowers' money balances. Recall that borrowers obtain money when they are young by selling labor-produced consumption good. The RHS is the demand for liquidity. It is the early-settling lender's unsettled IOUs. If settled when lenders are old, the face value of these debt contracts is $Af'\left(k_{t-1}^{*}\right)\kappa$, which is the capital income of the long-term production. If the supply of liquidity is abundant, these IOUs will sell at par value; that is, $\rho_t = 1$. Otherwise, IOUs are discounted and ρ_t is the market-clearing price. Thus, (15) holds with equality if and only if $\rho_t < 1$.

A competitive rational expectations equilibrium is defined as (i) borrowers and lenders maximize expected lifetime utility, taking prices as given; (ii) all markets clear; and (iii) the subjective distributions of production types and settling types are equal to the objective distribution of production types and settling types.

Proposition 1 *The equilibrium fraction of long-term producers is strictly positive, or* $1-\lambda_t > 0$ *. If* $\lambda_t > 0$*, both short-term producers and long-term producers choose* $k_t = k_t^{*} = \kappa$*.*

The intuition for Proposition 1 is as follows. Borrowers choose short-term technology because short-term output can be used to purchase unsettled IOUs in the secondary market. However, if all borrowers chose short-term technology, all IOUs would be paid when middle-age. In the secondary market, IOUs would be priced at par and there would be no profit. Thus, the returns from the long-term technology will dominate and some positive measure of borrowers will opt for the long-term technology.

In what follows, we focus on stationary equilibrium. With consumption good prices being the same across time, lenders will participate in the secondary market if they can. That is, $\tilde{\alpha}_t = \alpha$. Because $u(\cdot)$ is strictly concave and $g(\cdot)$ is strictly convex, the equilibrium is unique. Depending on the borrower's resources, we demonstrate that one of three possible equilibrium outcome will be obtained. The following conditions hold in all three cases.

$$
\begin{aligned}
g'(l_1^*) &= \frac{1}{\rho}u'(x_3^*) \\
\rho x_3^* &= l_1^* + \rho A\left[f(\kappa) - f'(\kappa)\kappa\right] \\
q_3^* &= Af'(\kappa)\kappa,\ q_2^* = 0 \\
q_2 + q_3 &= \rho A f'(\kappa)\kappa
\end{aligned}
$$

and $l_1 = l_1^*$, $x_3 = x_3^*$, $k = k^* = \kappa$.

The three cases identify three mutually exclusive partitions according to the fraction of short-term producers and the price of IOUs in the secondary market.

Case 1: $\lambda = 0$, $\rho = 1$, and $l_1^* \geq (1-\alpha) Af'(\kappa)\kappa$.

Case 2: $\lambda = 0$, $\rho = \frac{l_1^*}{(1-\alpha)Af'(\kappa)\kappa}$, and $(1-\alpha) f'(\kappa)\kappa \leq l_1^* < (1-\alpha) Af'(\kappa)\kappa$.

Case 3: $\lambda = \frac{(1-\alpha)f'(\kappa)\kappa - l_1^*}{f(\kappa)}$, $\rho = \frac{1}{A}$, and $l_1^* < (1-\alpha) f'(\kappa)\kappa$.

First, note that the debt settlement is endogenized in equilibrium. It is determined together with the discount price in the secondary market. Long-term producers produce l_1^* units of consumption goods when young in exchange for money. When money is ample in the secondary market as in case 1, all short-term IOUs are settled at par. Since there is no profit in the secondary market, all borrowers choose long-term technology. Early-settling lenders receive $(1-\alpha) Af'(\kappa)\kappa$, which is the capital income of long-term production. If money is scarce as in cases 2 and 3, short-term debts have to be discounted. If discount is mild, profit in the secondary market is not sufficient to induce short-term production (case 2). But if discount is big (case 3), some borrowers will choose short-term production. Because goods can settle debt contracts, the short-term technology output adds liquidity to the secondary market. In equilibrium, borrowers are indifferent to which technology to choose, because the return on the short-term production plus the profit in the secondary market equals the returns on the long-term production. Long-term production scales up the return to short-term production by A, the discount price is $1/A$, which is the lower bound on the discount price identified by case 3.

In cases 2 and 3, long-term producers have issued IOUs that exceed the value of their money holdings. Thus, both cases are marked by leveraging. Operationally, leveraging means that verifiable money is scarce and cannot be used to settle all IOUs in the short run. The liquidity problem arises when money is valued at a premium relative to the IOUs.

Timing mismatch, verifiability and leveraging are the three key ingredients that drive the liquidity problem in our model. Timing mismatch and the unverifiability of private debt contracts in the goods market create the secondary market for the resale of IOUs. However, timing mismatch and verifiability are not sufficient to generate liquidity problem (note that liquidity problem does not present in case 1). Borrowers have sufficient resources to avoid default. However, borrowers that are leveraged do not have resources available for the early-settling lenders as needed. In these scenarios, liquidity problem presents.

Our results provide a formal link between leveraging and liquidity problems in a general equilibrium setting. The recent literature on leveraging and liquidity has focused on principal-agent problems. For example, in Acharya and Viswanathan (2011), the borrower chooses the riskiness of the assets acquired with borrowed resources unbeknownst to the lender. Higher returns are associated with the higher risk assets. Such leveraging explains why negative shocks to asset values are associated with market liquidity problems. The negative shock to asset values means that the borrower is unable to repay their lenders or to rollover their debts.[9] In our setting, no hidden action problems exist between lenders and borrowers. The problem is not default risk, but settlement risk. Leveraging is present because borrowers can choose a technology that permits them to borrow more than the value of their middle-aged assets. This is not to say that principal-agent problems are not important. Rather, our view is that principal-agent problems are not necessary for leveraging to occur and therefore not necessary for leveraging to amplify liquidity problems into a crisis.

To further illustrate the three cases, consider a numeric example in which the measure of early-settling lenders is varied.

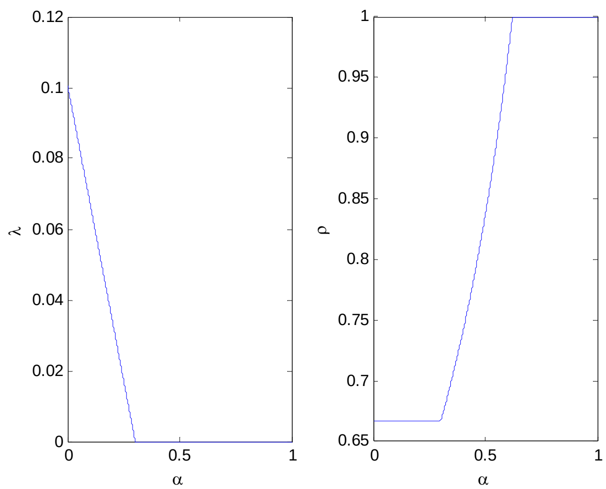
Example 1 *The utility functions for the lenders and borrowers are* $v(q_2+q_3) = \frac{(q_2+q_3)^{1-\sigma}}{1-\sigma}$ *and* $-g(l_1)+u(x_3) = \phi\sqrt{1-l^2} + \frac{x_3^{1-\sigma}}{1-\sigma}$*, respectively, where* $\sigma = 1.5$ *and* $\phi = 4$*. The production function is an intensive form of the Cobb-Douglas, where* $f(k) = k^{1/3}$*. The rest of the parameters are* $\kappa = 1$ *and* $A = 1.5$*. We vary* α *between* 0 *and* 1 *and plot* λ *and* ρ *as* α *changes (see Figure 1). In this exercise, case 1 occurs when* $0.62 < \alpha \leq 1$ *, case 2 occurs when* $0.32 \leq \alpha \leq 0.62$*, and case 3 occurs when* $0 \leq \alpha < 0.32$*.*

Figure 1

[9] See also, Adrian and Shin (2010). They identify leveraging by examining assets that are marked to market. They provide evidence that marked-to-market leveraging is procyclical.

Three Equilibrium Cases

Example 2



The numerical example illustrates the impact that different size liquidity shocks play in determining which equilibrium is realized. With $\alpha = 1$, for example, there is no liquidity shock because all lenders can participate in all markets. However, as α declines (i.e., the measure of early-settling lenders increases), the economy moves from equilibrium case 1, to case 2, and eventually case 3. That is, when the liquidity shock is large enough–a liquidity crisis can occur conditioned on the timing mismatch and leveraging. And the crisis is more serious when the liquidity shock is larger.

4 Planner's allocation

In the previous section, we identify the liquidity problem, leaving us to speculate on whether there exists a policy intervention that can improve welfare. Before we proceed to policy discussion, we first formulate a planner's allocation that characterizes the efficient stationary allocations. Each generation is given the same weight in the planner's objective function, though within each generation, the weights applied to lenders and borrowers are generalized. The planner faces no market frictions. Accordingly, the solution is the first-best allocation.

The planner maximizes weighted average utility of different types of agents subject to the resource constraints:

$$\max_{\substack{k,k^*,\lambda,l_1,l_1^*,\\ x_3,x_3^*,q_2,q_2^*,q_3,q_3^*}} \theta\left\{\lambda\left[-g\left(l_1\right)+u\left(x_3\right)\right]+\left(1-\lambda\right)\left[-g\left(l_1^*\right)+u\left(x_3^*\right)\right]\right\}+\left(1-\theta\right)\left[\left(1-\alpha\right)v\left(q_2+q_3\right)+\alpha v\left(q_2^*+q_3^*\right)\right] \tag{16}$$

$$s.t.\ \lambda\left[l_1+f\left(k\right)\right]+\left(1-\lambda\right)\left[l_1^*+Af\left(k^*\right)\right]=\left(1-\alpha\right)\left(q_2+q_3\right)+\alpha\left(q_2^*+q_3^*\right)+\lambda x_3+\left(1-\lambda\right)x_3^* \tag{17}$$

$$\kappa=\lambda k+\left(1-\lambda\right)k^* \tag{18}$$

$$0\leq\lambda\leq 1 \tag{19}$$

where θ is the weight given to borrower's lifetime in the planner's welfare function. The first two constraints are the resource constraints for consumption good and capital good, respectively. The third constraint indicates that the fraction of short-term producers cannot lie outside the unit interval.

Let the "hat" on the variables denote the values that satisfy the first-order conditions for the planner's problem. After simplifying, planner's allocation is characterized by the following equations:

$$\hat{\lambda} = 0 \tag{20}$$

$$\hat{k}^* = \kappa \tag{21}$$

$$\hat{q}_2+\hat{q}_3 = \hat{q}_2^*+\hat{q}_3^* \tag{22}$$

$$u'\left(\hat{x}_3^*\right) = \frac{1-\theta}{\theta}v'\left(\hat{q}_2^*+\hat{q}_3^*\right) \tag{23}$$

$$g'\left(\hat{l}_1^*\right) = \frac{1-\theta}{\theta}v'\left(\hat{q}_2+\hat{q}_3\right) \tag{24}$$

Note that the planner's allocation is not unique because lenders treat middle-age and old-age consumption as perfect substitutes. Without loss of generality, we assume $\hat{q}_2=\hat{q}_2^*=0$. That is, young borrowers only produce for the old. Because long-term technology dominates the short-term technology, the planner allocates the economy's entire capital endowment to long-term production. The planner's allocation provides full risk sharing to lenders.

By comparing the first-order conditions in the planner's allocation and three cases characterizing equilibrium in the decentralized economy, we find the planner's allocation is implemented in the decentralized economy in Case 1, but not in Cases 2 or 3.

Proposition 2 *The stationary equilibrium in the decentralized economy achieves the planner's allocation for some welfare weight θ if and only if the equilibrium is in case 1.*

5 Repurchase agreements and transfers

Suppose there is a central bank that operates in debt settlement and the secondary market. How would the central bank solve the liquidity problem in cases 2 and 3 of the decentralized economy so that the planner's allocation is implemented? Consider a three-step mechanism. First, the central bank conducts an open market purchase in the secondary market, offering $\tilde{m}$ units of fiat money in exchange for private debt contracts at a discount price ρ. In the next period, IOUs are settled by long-term producers, offering goods as payment. The central bank then sells the goods to old agents such that the nominal value of the goods is equal to $\tilde{m}$ units of fiat money. Due to the overlapping generations structure of the model, the money injected to the secondary market (to purchase IOUs issued by generation t) is removed from the economy by the money taken out from the goods market (for the goods consumed by generation $t-1$) in the same period. The transactions between the central bank and lenders are a repurchase agreement–the central bank purchases private IOUs from the middle-aged early-settling lenders using money and repurchases money using consumption goods in the next period. Of course, $\tilde{m}$ and ρ need to be set to clear the resale market.

In addition to the repurchase agreement, the central bank operates a lump-sum tax and transfer process. Let $T_B, T_{B^*}, T_L,$and T_{L^*} denote the net life-time tax on the short-term producers, long-term producers, early-settling lenders, and late-settling lenders, respectively. The only restriction is that young lender's capital is not taxable.[10] The central bank runs a balanced budget in each period. The following proposition is derived with these policy tools.

Proposition 3 *The optimal price of IOUs in the secondary market is par, which implements planner's allocation in the decentralized economy.*

Proof. Let $\rho = 1$, $T_B > T_{B^*}$, and $T_L = T_{L^*}$. The central bank commits to buy unlimited amount of private IOUs at the price of $\rho = 1$ (although in equilibrium only a finite amount of money is

[10] This assumption is to make sure that markets do not disappear with the tax scheme. The central bank cannot simply force lenders to give up all capital when young and compensate them with consumption goods when middle-aged and old.

injected). Because this policy action renders zero profits in the secondary market and because long-term producers are taxed less, there are only long-term producers in equilibrium. With $\rho = 1$, the borrower's first-order condition is identical to the first-order condition in the planner's problem and the lender's consumption is the same regardless whether they are permitted to meet in the settlement and secondary markets once or twice. As the central bank commits to provide unlimited liquidity, the liquidity supply in the secondary market is sufficient and (15) is irrelevant for the equilibrium. Lastly, the value of T_{B^*} and T_{L^*} are chosen in such a way that the marginal utility for the two types of agents satisfies (23) (or (24)). The total value of liquidity injected in the secondary market is $\frac{\tilde{m}}{p_x} = (1-\alpha) A f'(\kappa) \kappa - \hat{l}_1^*$. ■

Policies can implement the planner's allocation because lenders are indifferent between consuming when middle-aged or old-aged. In other words, lenders treat short-term output and long-term output as perfect substitutes. The timing mismatch is resolved by permitting the central bank to serve as holder of private IOUs. When the goods are available, the holder applies these consumption goods to redeeming IOUs held by early-settling lenders. Consequently, the optimal allocation is to invest all capital to higher-return, long-term production. Because the IOUs are sold at par under the central bank policy, the profit in the secondary market is zero. All borrowers choose the long-term production. The inefficiency present in the decentralized economy is purely a liquidity problem. The central bank overcomes production inefficiency by replacing a cheaply provided good–money–instead of using a costly one–short-term production.

Note that our repurchase agreement is a modified version of the unconventional monetary policy put forward in Gertler and Karadi (2011). In our setup, the central bank directly purchases private credit instruments. In their paper, Gertler and Karadi demonstrate how unconventional monetary policy can moderate the amplitude of a cyclical fluctuation by injecting liquidity. Here, we show three things related to unconventional monetary policy. First, stationary output levels are greater when the unconventional monetary policy is implemented compared with when there is no active monetary policy. Second, one-period and two-period returns are equalized for lenders; in other words, the yield curve flattens. Third, the unconventional monetary policy deals with distribution of liquidity not the aggregate quantity of liquidity.

In addition, unconventional monetary policy offers a different perspective on leveraging and liquidity. The existing literature has focused the principal-agent characteristics that induce leveraging, thereby ignoring solutions to the liquidity problems that arise. In particular, the literature does not address whether an agent or institution exists capable of choosing policies that implement the efficient allocation. Our setup treats the central bank as a third-party institution controlling the amount and the distribution of liquidity. When facing a liquidity problem, the central bank responds by creating liquidity based on the value of the underlying asset. Since there is no default risk, the central bank purchases private IOUs at par. When it matures, IOUs are settled and liquidity is extracted. In this way, any transfers are undone.

6 An economy with generation-specific consumption goods

We modify the model to consider an economy in which the commodity space consists of generation-specific consumption goods and physical capital. Capital endowed with generation t produces goods that are valued only by their own generation. The reason why we introduce generation-specific goods is that borrowers may have specific use of the loan. The output from the loan may only be valued by a particular pair of lender-borrower. Here, we assume borrowers and lenders in each generation is a particular pair. Meanwhile, we want to keep the use of money to be universal. So we assume that labor is a general input. That is, young borrowers can produce any generation-specific goods for the middle-aged and old agents.

The planner's objective function is the same as in (16). The resources constraint for consumption good is changed. Because in every period there are two generation-specific goods, the planner decides how much of each generation-specific good needs to be allocated. Formally, the resource constraints for consumption goods are

$$\lambda \pi l_1 + (1-\lambda)\pi^* l_1^* + \lambda f(k) = (1-\alpha) q_2 + \alpha q_2^* \tag{25}$$

$$\lambda (1-\pi) l_1 + (1-\lambda)(1-\pi^*) l_1^* + (1-\lambda) A f(k) = (1-\alpha) q_3 + \alpha q_3^* + \lambda x_3 + (1-\lambda) x_3^* \tag{26}$$

Equations (25) and (26) are the constraints for the consumption goods for the middle-aged lenders and the old-aged lenders and borrowers, respectively, where π and π^* are the fraction of short-term and long-term producer's labor devoted to the production of goods consumed by middle-aged lenders. The resource constraint for capital good remains unchanged.

It is easy to verify that with $\pi = \pi^* = 0$, the solution to the planner's problem in section (4) also solves the planner's problem here.[11] The intuition is simple. Because lenders view middle-age consumption and old-age consumption as perfect substitutes and borrowers only value old-age consumption, the planner can simply allocate all resources to long-term production and agents consume when old. By doing this, the planner takes full advantage of the higher-return technology to maximize total output and consumption.

In the decentralized economy, the prices of consumption goods need to be indexed by generations. Young borrowers have the choice to produce for middle-aged or old agents, and they will choose to produce the goods with higher price. Therefore, $m_t = \max\left\{p_{xt}^{t-1}, p_{xt}^{t-2}\right\} l_t$, where the superscript on p denotes which generation consumes the goods. Similarly, $m_t^* = \max\left\{p_{xt}^{t-1}, p_{xt}^{t-2}\right\} l_t^*$. The prices appear in lender's budget constraints (i.e., equations (1)and(2)) and old borrower's budget constraints (i.e., equations (7) and (11)) are indexed by their own generation. In each period, two consumption good markets open at the same time. At time t, the market clearing conditions for goods consumed by generations born in $t-1$ and $t-2$, respectively, are

$$\begin{aligned}
\lambda_t \pi_t l_t + (1-\lambda_t)\pi_t^* l_t^* + \lambda_{t-1} f(k_{t-1}) &= (1-\tilde{\alpha}_{t-1}) q_{2t} + \tilde{\alpha}_{t-1} q_{2t}^* \\
\lambda_t (1-\pi_t) l_t + (1-\lambda_t)(1-\pi_t^*) l_t^* + (1-\lambda_{t-2}) Af\left(k_{t-2}^*\right) &= \lambda_{t-2} x_{3t} + (1-\lambda_{t-2}) x_{3t}^* + \\
&\quad (1-\tilde{\alpha}_{t-2}) q_{3t} + \tilde{\alpha}_{t-2} q_{3t}^*
\end{aligned}$$

Again, we focus on stationary equilibrium. In particular, we focus on the equilibrium in which consumption goods, regardless of their generation types, have the same prices in all periods. It can be showed that the three cases in section (3) hold here. It follows that the central bank buys private IOUs at face value, thus achieving the planner's allocation in the decentralized economy. The introduction of generation-specific goods does not change any of the results obtained in the model economy with a single consumption good. However, as we will see in the next section, we modify preferences and the results are different.

[11] Note that the planner's allocation is not unique – π and π^* can be positive as lender's middle-age and old-age consumption are perfect substitutes and allocating young borrower's labor to producing two different goods does not cause extra social costs. Without loss of generality, we can assume $\pi = \pi^* = 0$ here.

7 A Fisherian lender

In this section, we keep our assumption of generation-specific goods, and we modify the early-settling lender's preference. The late-settling lender's utility function remains as before. Let the utility function of the early-settling lenders be $v(q_{2t+1})$.[12] Hence, the early-settling lenders have to settle when middle-aged because they do not value consumption received when old at all. In this environment, short-term goods and long-term goods are no longer perfect substitutes for the early-settling lenders. And because goods are generation-specific, the early-settling lender's consumption cannot be provided by the long-term production that started two-periods before. Therefore, the preference shocks may impose a cost as some resources need to be allocated in the lower-return technology.

7.1 Planner's allocation

The planner now maximizes the weighted average utility with the modified utility function of the early-settling lenders, subject to the resource constraints $(18)-(19)$ and $(25)-(26)$ with $q_3 = 0$. With the change in preference, the planner's solution now has three cases. In all cases, borrowers are treated equally. Without loss of generality, we assume $q_2^* = 0$.

Planner Allocation I: $\hat{\lambda} = 0$, $\hat{\pi}^* < 1$. All capital is allocated in long-term technology. Early-settling lenders consume labor-produced goods. The labor-produced output is sufficient enough to satisfy the consumption of the early-settling lenders in the sense that full risk sharing is achieved. Equations $(21)-(24)$ hold here with $\hat{q}_3 = 0$. The consumption allocation is the same as in the baseline model.

Planner Allocation II: $\hat{\lambda} = 0$, $\hat{\pi}^* = 1$. All capital is allocated in long-term technology. The planner rations the output of the young borrowers among the early-settling lenders. Early-settling lenders consume strictly less than the late-settling lenders. That is, $\hat{q}_2 < \hat{q}_3^*$. The planner's solution is described by equations (23) and (24) with $\hat{q}_3 = 0$, and the resource constraints. It follows that $\frac{1}{A} < \frac{u'(\hat{x}_3^*)}{g'\left(\hat{l}_1^*\right)} < 1$. This case occurs when the marginal social cost of having 1 extra unit of capital placed in the lower-return technology is higher than the marginal social benefit of having early-settling lenders consume $f(1)$ extra units of the generation-specific consumption goods.

[12] This preference shock is the same as in Diamond and Dybvig (1983).

Planner Allocation III: $\hat{\lambda} > 0$, $\hat{\pi}^* = \hat{\pi} = 1$. The planner allocates capital in short-term and long-term production. This case occurs when the marginal social cost equates the marginal social benefit of having one unit of capital in the short-term technology. The allocation is described by

$$g'\left(\hat{l}_1^*\right) = Au'\left(\hat{x}_3^*\right),$$

equations (23) and (24) with $\hat{q}_3 = 0$, and the resource constraints. Again, $\hat{q}_2 < \hat{q}_3^*$ holds in this case.

7.2 Repurchase agreement and transfer policies

The equilibrium in the decentralized economy remains the same given the modification of the lender's preference. However, by changing the lender's preferences, we derive very different results in terms of the equilibrium efficiency and its implication for the monetary policy. The results are presented in the following proposition.

Proposition 4 *Equilibrium case 1 implements Planner Allocation I. With a proper lump-sum tax scheme, equilibrium cases 2 and 3 implement Planner Allocation II and III, respectively.*

Proof. See appendix. ■

The first part of the proposition is the same as in Proposition 2. With ample liquidity, all capital is devoted to long-term production and full risk sharing is achieved between two different types of lenders in equilibrium.

The second part of the proposition says that with the change in preference, the equilibrium in case 2 and case 3 is efficient even without the unconventional monetary repurchase agreement. In both cases, the discount price is less than 1. This price reflects the fact that the middle-age consumption is more costly because the resource for middle-age consumption is more scarce (in case 2 the consumption comes from young borrower's production only) and it requires some resources invested in a lower-return technology (as in case 3). The lump-sum tax scheme is needed to achieve the desired level of risk sharing among lenders. It does not drive wedges between the prices and the marginal rate of substitutions of the borrowers.

The changes in results and policy implications in our modified model tells us how important the identification of the friction is. The market price reflects lender's marginal cost of consuming when middle-aged versus when old. However, the market does not distinguish the driving force of this cost–whether it is due to a purely timing friction imposed by the market participation restriction or due to a preference shock. When it is due to a timing friction, the discount price is the liquidity premium of money. So injecting liquidity can solve the problem and the unconventional repurchase agreements are optimal. When it is due to a preference shock, the discount price reflects the fundamental cost of middle-age consumption. Monetary intervention is not necessary and it can even harm the economy as it distorts the producer's marginal rate of substitution. With a slight modification to preferences, the identification is more challenging. Simply put, the price of IOUs is not sufficient to indicate a liquidity problem that exists concomitantly with an inefficient equilibria.

8 Conclusion

We build a model to capture many features associated with financial crises. Perhaps most importantly, leveraging plays a key role in the set of conditions necessary for a liquidity crisis to occur. Indeed, we show that the equilibrium in our baseline model is inefficient only if the borrower has leveraged their capital purchases such that the value of the debts exceeds the value of fiat money holdings. We also focus on events in secondary asset markets and debt settlements.[13] Our central question is, what is the optimal policy response to the liquidity crisis? In our setup, the liquidity problem is not a shortage of aggregate liquidity, it is the distribution of liquidity that drives the inefficiency. An unconventional repurchase agreement implements the efficient allocation by redistributing liquidity across different markets.

We demonstrate that unconventional monetary policy depends on the types of underlying frictions that drive the price in the secondary market away from the par. The underlying frictions are not perfectly identifiable by discounts in secondary markets. In the baseline model, the idiosyncratic shock is modelled as a timing mismatch in debt settlements that owes to a settlement participation risk. The monetary policy is effective because it eliminates the timing friction by substituting the central bank for the lenders temporarily in debt settlements. However, if the timing mismatch is driven by preference shocks as in Diamond-Dyvbig (1983), the apparent fire sale price embodies the higher cost of consuming early. The unconventional monetary policy is not able to address such a

[13] Gorton (2007) submits that activity in secondary markets were important in the emergence of the most recent liquidity crisis that started in 2007.

real shock. Hence, the application of unconventional monetary policy requires careful examination of the nature of the shock.

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Appendix

Proof of lemma 1: We prove the lemma by checking the lender's budget constraints.

Prove by contradiction. First consider $a_t < 1$. Suppose that $\rho_{t+1}(1+\gamma_t) > 1$. The lender's budget sets expand as a_t shrinks. This is true for both early-settling and late-settling lenders. The young lender prefers to lend only to the long-term producers since the product of the discounted price and the long-term loan markup exceeds the face value of the loan. With such an arbitrage opportunity, lenders will lower the interest rate to $\gamma_t - \varepsilon_1$, where $\varepsilon_1 \to 0$ and raise the interest rate on short-term loan to $1 + \varepsilon_2$, where $\varepsilon_2 \to 0$. This continues until the arbitrage opportunity vanishes. The upshot is that $1 + \gamma_t > 1/\rho_{t+1}$ cannot be the equilibrium interest rate on long-term production loan.

Alternatively, suppose that $\rho_{t+1}(1+\gamma_t) < 1$. In this case, both early-settling and late-settling lenders see their budget sets expand as a_t increases. In this case, a lender prefers to raise the long-term interest rate, thus excluding long-term producers. It follows that $1 + \gamma_t < 1/\rho_{t+1}$ cannot be the equilibrium interest rate on long-term production loan.

Lastly, consider $a_t = 1$. All IOUs are settled in the short term. The discount price is 1 because the IOUs will be settled at face value when the date$-t$ generation is middle-aged.

Proof of proposition 1: First, prove $\lambda_t < 1$ by contradiction. Consider an equilibrium with $\lambda_t = 1$. It follows that borrowers choose only short-term production if and only if lifetime utility is higher when calculated under short-term production than under long-term production. With no long-term producers, all IOUs are settled, rendering the secondary market moot and $\rho_{t+1} = 1$. So, we need to show that at least one borrower would realize higher lifetime utility by choosing long-term production.

When satisfied, equations (9) and (13) imply that $f'(k_t) = Af'(k_t^*) = p_{kt}/p_{xt+1}$. The strictly concavity of the production function implies that $k_t < k_t^*$ for this case. After rearranging, the short-term producer's lifetime budget constraint is written as

$$f(k_t) - f'(k_t)\, k_t = -\frac{p_{xt}}{p_{xt+2}} l_{1t} + x_{3t+2}$$

while the long-term producer's lifetime budget constraint is

$$A\left[f(k_t^*) - f'(k_t)\, k_t^*\right] = -\frac{p_{xt}}{p_{xt+2}} l_{1t}^* + x_{3t+2}^*.$$

It is sufficient, therefore to compare $f(k_t) - f'(k_t)\,k_t$ and $A\,[f(k_t^*) - f'(k_t^*)\,k_t^*]$ to determine which producer type has the larger budget set. After rearranging, $A\,[f(k_t^*) - f'(k_t^*)\,k_t^*] - [f(k_t) - f'(k_t)\,k_t] = Af(k_t^*) - f(k_t) - Af'(k_t^*)\,(k_t^* - k_t)$. With $f(k)$ strictly concave, it follows that

$$Af(k_t^*) - Af'(k_t^*)\,(k_t^* - k_t) > Af(k_t)\,.$$

Next, subtract $f(k_t)$ from both sides, yielding $Af(k_t^*) - f(k_t) - Af'(k_t^*)\,(k_t^* - k_t) > Af(k_t) - f(k_t)$ where $Af(k_t) - f(k_t) > 0$ for $A > 1$. Thus, the long-term producer's budget set is bigger, implying that the long-term producers' lifetime utility is strictly greater than the short-term producer's lifetime utility. A contradiction.

If $1 - \lambda_t > 0$, then short-term production and long-term production coexist. It follows that lifetime welfare must be the same for the two types. As their marginal rates of intertemporal substitution are equal, their life-time budget constraints are the same. As such, we can write,

$$f(k_t) - f'(k_t)\,k_t = \rho_{t+1} A\left[f(k_t^*) - f'(k_t)\,k_t^*\right] \tag{27}$$

Combine with (9) and (13) to get

$$\frac{f(k_t)}{f'(k_t)} - k_t = \frac{f(k_t^*)}{f'(k_t^*)} - k_t^*. \tag{28}$$

Because $f(k)/f'(k) - k$ is monotone by concavity of $f(\cdot)$, we have $k_t = k_t^*$. Further, the capital market-clearing condition guarantees that $k_t = k_t^* = \kappa$.

Proof of Proposition 3 The equilibrium with tax scheme can be characterized by the following equations. On the borrower's side, the budget constraints and the first-order conditions (combined with capital market clearing conditions) that describe the decision of (l_1^*, x_1^*) are

$$Af(\kappa) + \frac{l_1^* - T_{B^*,1} - f'(\kappa)\,\kappa}{\rho} - T_{B^*,3} \quad = \quad x_3^* \tag{29}$$

$$g'(l_1^*) - \frac{1}{\rho} u'(x_3^*) \quad = \quad 0 \tag{30}$$

where $T_{B^*,1}$ and $T_{B^*,3}$ are the taxes that the borrowers pay when young and old, respectively. Note that short-term producers, if there is any, will be taxed of the same amount as the long-term producers to achieve the social optimum.

The lenders are taxed so that their consumption is

$$q_2 \quad = \quad \rho\left[Af'(\kappa)\,\kappa - T_{L,3}\right] - T_{L,2} \tag{31}$$

$$q_3^* \quad = \quad Af'(\kappa)\,\kappa - T_{L^*} \tag{32}$$

where $T_{L,2}$ is the transfer to the middle-aged lenders and $T_{L,3}$ is the tax when they are old. Furthermore, let $T_{L,2} \leq 0$ and $T_{L,3} \geq 0$. That is, we tax early-settling lenders when they are old but not when they are middle-aged so no there is no production in the short-term production will be wasted.

The secondary market clearing condition is

$$l_1^* - T_{B^*,1} + \lambda f(\kappa) \geq (1-\alpha)\rho\left[Af'(\kappa)\kappa - T_{L,3}\right] \tag{33}$$

The balanced budget constraint is

$$T_{B^*,1} + T_{B^*,3} + (1-\alpha)(T_{L,2} + T_{L,3}) + \alpha T_{L^*} = 0 \tag{34}$$

The feasibility constraint for early-settling lenders tax/transfer is

$$T_{B^*,1} \geq -(1-\alpha) T_{L,2} \tag{35}$$

To achieve $\left(\hat{l}_1^*, \hat{x}_3^*, \hat{q}_2, \hat{q}_3^*\right)$ in Planner Allocation II, the lump-sum tax scheme can be set as follows. Tax short-term producers sufficiently high so there is no short-term producers in equilibrium so $\lambda = 0$. Let

$$\begin{aligned}
T_{B^*,1} + \frac{u'(\hat{x}_3^*)}{g'\left(\hat{l}_1^*\right)} T_{B^*,3} &= \frac{u'(\hat{x}_3^*)}{g'\left(\hat{l}_1^*\right)} Af(\kappa) + \hat{l}_1^* + f'(\kappa)\kappa - \frac{u'(\hat{x}_3^*)}{g'\left(\hat{l}_1^*\right)} \hat{x}_3^* \\
T_{B^*,1} - (1-\alpha)\frac{u'(\hat{x}_3^*)}{g'\left(\hat{l}_1^*\right)} T_{L,3} &= \hat{l}_1^* - (1-\alpha)\frac{u'(\hat{x}_3^*)}{g'\left(\hat{l}_1^*\right)} Af'(\kappa)\kappa \\
T_{B^*,1} + (1-\alpha) T_{L,2} &= 0
\end{aligned}$$

and equation (34). Note that the number of unknowns $(T_{B^*,1}, T_{B^*,3}, T_{L,2}, T_{L,3}, T_{L^*})$ exceeds the number of equations, so the conditions that $T_{L,2} \leq 0$ and $T_{L,3} \geq 0$ can always be satisfied. Given the tax scheme, the solution to $(l_1^*, x_3^*, q_2, q_3^*, \rho)$ are solved by (29) – (33) with (33) holds with equality and $\lambda = 0$. By convexity of the problem, the solution is unique and the same as the planner's allocation.

To achieve $\left(\hat{l}_1^*, \hat{x}_3^*, \hat{q}_2, \hat{q}_3^*\right)$ in Planner Allocation III, the lump-sum tax scheme can be set as follows. Tax short-term producers the same amount as long-term producers when young and old. That is, $T_{B,1} = T_{B^*,1}$ and $T_{B,3} = T_{B^*,3}$ The short-term producer's life-time budget constraint is

$$\frac{f(\kappa) + l_1 - T_{B,1} - f'(\kappa)\kappa}{\rho} - T_{B,3} = x_3 \tag{36}$$

With the same tax obligation, short-term producers and long-term producers have the same budget constraint iff $\rho = 1/A$.

Let

$$\begin{aligned} T_{B^*,1} + \frac{1}{A} T_{B^*,3} &= f(\kappa) + \hat{l}_1^* + f'(\kappa)\kappa - \frac{1}{A}\hat{x}_3^* \\ T_{B^*,1} - (1-\alpha)\frac{1}{A} T_{L,3} &= \hat{l}_1^* + \hat{\lambda} f(\kappa) - (1-\alpha) f'(\kappa)\kappa \\ T_{B^*,1} + (1-\alpha) T_{L,2} &= 0 \end{aligned}$$

and equation (34). Again, the number of unknowns exceeds the number of equations and the conditions that $T_{L,2} \leq 0$ and $T_{L,3} \geq 0$ can always be satisfied. Given the tax scheme, the solution to $(l_1^*, x_3^*, q_2, q_3^*, \lambda)$ are solved by (29)−(33) with (33) holds with equality and $\rho = 1/A$. By convexity of the problem, the solution is unique and the same as the planner's allocation.