Money Creation and Banking: Theory and Evidence*

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Abstract

This paper develops a monetary-search model where the money multiplier is endogenously determined. I show that when the central bank pays enough interest on reserves, the money multiplier depends on the nominal interest rate and the interest on reserves rather than the reserve requirement. The calibrated model can explain the evolution of the money multiplier and the excess reserve-deposit ratio in the pre-2008 and post-2008 era. Moreover, I find that the dramatic changes in the money multiplier after 2008 are driven by the introduction of the interest on reserves with the low nominal interest rate.

JEL Classification Codes: E42, E51

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> [M1] is a sum of currency holdings that do not pay interest and demand deposits that (in some circumstances) do. ... a model of the banking system in which currency, reserves, and deposits play distinct roles ... seems essential if one wants to consider policies like reserves requirements, interest on deposits, and other measures that affect different components of the money stock differently. **Lucas (2000)**

1 Introduction

Since the Great Recession, the conduct of U.S monetary policy has changed substantially. The federal funds rates remained near zero during 2009-2016, and the Federal Reserve has been paying interest on reserves since October 2008. For the post-2008 era, one of the important features of the monetary policy is a dramatic decline in the M1 to monetary base ratio (M1 money multiplier, hereafter) and a large increase in excess reserves. The M1 money multiplier fell in half from 1.6 to 0.8, and the excess reserves to checkable deposit ratio increased from 0 to 1.4 during 2007-2015.

The first goal of this paper is to build a model that endogenously determines bank's demand for reserves and its creation of inside money given the availability of different means of payments. To that end, I extend the search-theoretic monetary model of Lagos and Wright (2005) by introducing fractional reserve banking and unsecured credit. Most of the monetary models have been silent on how the money multiplier is determined when banks are holding excess reserves. The textbook models of money creation only focus on the case where banks lend until they have zero excess reserves. Since the banks have been holding excess reserves since 2008, natural questions that arise are: Why are banks holding excess reserves but they did not before 2008? What drives the huge drop in the money multiplier? The main contribution of my papers is to study why banks are holding excess reserves and the money multiplier are dropped since 2008.

The decrease in the money multiplier is not is a recent phenomenon. To illustrate this, the top-left panel of Figure 1 shows that the M1 money multiplier has been decreasing since 1990. However, these changes are not closely related to the required reserve ratio.[1] To illustrate this, the bottom-left panel of Figure 1 plots the money

[1]The required reserve ratio and currency deposit ratio presented in Figure 1 are computed by (Required Reserves)/(Total Checkable Deposits) and (Currency Component of M1)/(Total Checkable Deposits). The legal reserve requirement for net transaction accounts has been 10% from April 2, 1992 to March 25, 2020, but some banks are imposed upon by lower requirements or exempt depending on the size of their liabilities. These criteria have been changed 27 times from the 1st quarter 1992 to the last quarter of 2019. From March 2020, all the required reserve ratios became zero. See Feinman (1993) and https://www.federalreserve.gov/monetarypolicy/reservereq.htm for more details on the

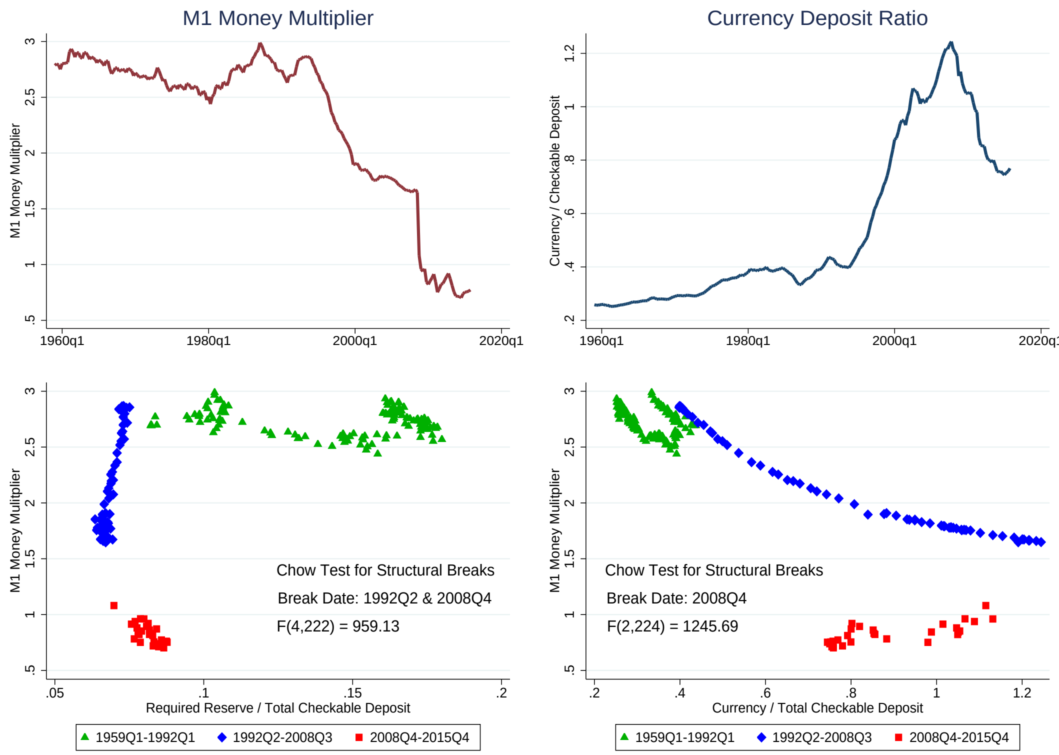

Figure 1: Money multiplier, currency deposit ratio and required reserve ratio

Chow tests for structural breaks are implemented with null hypothesis of no structural break. Appendix B contains details of the Chow tests.

multiplier against the required reserve ratio. While the excess reserve ratio had been close to zero until 2008 as shown in Figure 2, the bottom-left panel of Figure 1 does not show any negative relationship between the M1 money multiplier and required reserve ratio. The absence of a correlation suggests that required reserve ratio do not drive the money multiplier in pre-2008 period. According to the model, the decline in the money multiplier during pre-2008 period can be attributed by changes in credit condition.

Another feature of the economy during this time period is the steady increase in the demand for physical currency, despite the availability of electronic payments. The currency to GDP ratio rose from 4% in 1990 to 7.3% in 2015. Further, the U.S. economy had more physical currency than checkable deposits from the 2nd quarter of 2002 to the 1st quarter of 2010. The top-right panel of Figure 1 shows that the currency to deposit ratio increased dramatically since 1990 but fell after 2008. The bottom-right panel of Figure 1 shows a strong negative relationship between currency deposit ratio and money multiplier until the 3rd quarter of 2008 and disappeared since the Federal Reserve started paying interest on reserves. My paper contributes

historical evolution of reserve requirement policy of the United States.

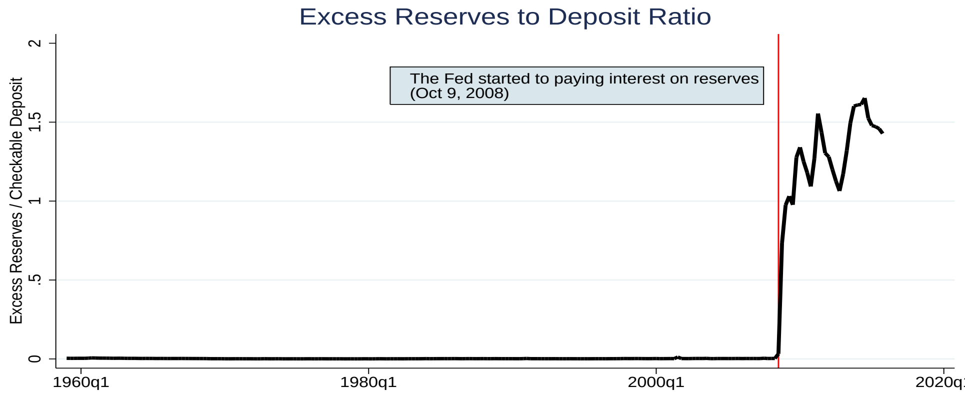

Figure 2: Excess reserves to deposit ratio

to our understanding of how the Federal Reserves' new approach to monetary policy has affected to the relationship between currency deposit ratio and money multiplier since the Great Recession. Prior research (e.g. Freeman and Huffman, 1991; Freeman and Kydland, 2000) before 2008 found negative relationship between the currency-to-deposit ratio and the money multiplier. However, there is no such relationship anymore and I show that the interest on reserves plays an important role for this change.

The model includes the explicit structure of monetary exchange and the role of financial intermediation. Buyers an sellers can trade by using cash, claims on deposits (e.g., check or debit card), banknotes, and unsecured credit (e.g., credit card). The bank creates banknotes by making loans and its lending is constrained by the reserve requirement. The equilibrium is in one of the three cases: a scarce-reserves equilibrium, an ample-reserves equilibrium and a no-banking equilibrium.

I find that the *scarce-reserves equilibrium* arises when the nominal interest rate is sufficiently high. In the scarce-reserves equilibrium, the bank's lending limit binds. If the central bank lowers the nominal interest rate, the reserve balance increases. The bank creates banknotes proportional to reserves. Since the reserve requirement determines this proportion, the reserve requirement affects the money multiplier.

The *ample-reserves equilibrium* occurs when the central bank pays interest on reserves and sets the nominal interest rate at some moderate level. In this case, the bank holds excess reserves. In the ample-reserves equilibrium, the bank's lending limit does not bind. The reserve requirement does not change the money multiplier. Instead the money multiplier is determined by the nominal interest rate and the interest on reserves. Lowering the nominal interest rate increases reserves but the bank does not create banknotes proportionally, which lowers the money multiplier. A higher interest on reserves decreases the money multiplier since the bank has more incentive to hold reserve and less incentive to create banknotes.

The *no-banking equilibrium* arises if the nominal interest rate is low enough. In the no-banking equilibrium, there is sufficient outside money to trade so that the buyers do not need to deposit their balances to the bank. The bank can not create the inside money because it does not hold any reserves.

The interaction between unsecured credit and other means of payment is in line with Gu, Mattesini and Wright (2016) and Lester, Postlewaite and Wright (2012). Some fraction of agents can trade using unsecured credit, and real balances respond endogenously as the credit condition changes. Better credit conditions reduce the real balance of inside money and reserves, but not cash, which results in a lower money multiplier.

The second goal of this paper is to quantify the model to determine the impact of the monetary policy and the introduction of consumer credit on reserves and money multiplier. The model is parameterized to match the pre-2008 U.S data. Quantitatively, the calibrated model can account for the behavior of money creation before and after 2008. The model generated series can mimic the historical behavior of the M1 money multiplier, the excess reserves to deposit ratio, and the currency deposit ratio. The welfare analysis shows that lowering reserve requirement or paying interest on reserves can reduce the welfare cost of inflation. Also, the quantitative analysis identifies the source of changes in the money multiplier and means of payment. The counter-factual analysis shows that a pre-2008 trend of decreasing money multiplier is driven by an increase in unsecured credit while a post-2008 trend of decreasing money multiplier is not attributed to the increase in unsecured credit. From the model and data, I provide evidence that suggests the dramatic changes in the money multiplier after 2008 are mainly driven by the Federal Reserve's monetary policy: the introduction of the interest on reserves with the low nominal interest rate.

Related Literature This work builds on three branches of literature. First and foremost, this work relates to the literature that explicitly studies money, credit, and financial intermediation to understand monetary transmission. As competing means of payment, it is important to understand the interaction between money and credit. There are many ways to introduce credit to the monetary economy. For example, Sanches and Williamson (2010) study the environment where money and credit are competing means of payment due to imperfect memory, limited commitment, and theft. Lotz and Zhang (2016) develop a model of money and credit where sellers must invest in record-keeping technology to accept credit. Williamson (2016) introduces a model of banking that allows agents to use money and asset-backed credit, and Berentsen, Camera and Waller (2007) introduce banks as financial intermediaries so that they accept deposits and issue IOUs with an enforcement technology. Gu et al. (2016) show that credit is a substitute for money and it crowds out the real balance of money. This paper follows Gu et al. (2016) and Berentsen et al. (2007) in the sense that the model allows agents to use unsecured debt with an exogenous credit limit and banks to issue private IOUs using its enforcement technology.

Second, this paper contributes to a large literature on inside money and money creation. The difference between inside money and outside money has been highlighted since Sargent and Wallace (1982) and Freeman and Huffman (1991). Previous works capturing the explicit role of reserve requirements include Freeman (1987), Haslag and Young (1998), and Freeman and Kydland (2000). Freeman (1987) and Haslag and Young (1998) study the impact of money creation and reserve requirements on seigniorage revenue. Freeman and Kydland (2000) develop a tractable model of an endogenous money multiplier. They show that the money-output correlation can be explained by the endogenous money supply resulting from households' choices in response to the business cycle. Recent advances in monetary economics based on search-theoretic framework provide a deeper understanding on banking and inside money. For example, in the Lagos and Wright (2005) framework, Gu, Mattesini, Monnet and Wright (2013) study the environment where banking arises endogenously, and they show that banking can improve the economy by facilitating trade using inside money. Andolfatto, Berentsen and Martin (2018) integrate the Diamond (1997) model of bank and financial markets into the Lagos and Wright (2005) framework and deliver a model where the fractional reserve banking arises in the equilibrium. This paper contributes to the literature by constructing a monetary-search model of banking to explain the money creation process.

Finally, this paper contributes to the literature that studies monetary policy with interest on reserves. Many previous studies, like Cochrane (2014), are based on the asset pricing framework by focusing on reserves as safe interest-bearing assets. Some recent works (e.g., Williamson, 2018; Armenter and Lester, 2017; Afonso, Armenter and Lester, 2019) study the role of interest on reserves in the interbank market and on monetary policy implementation. While previous works do not focus on the role of interest on reserves in the inside money creation process, this paper shows that interest on reserves plays a crucial role in the inside money creation process.

This paper is organized as follows. Section 2 constructs the search-theoretic monetary model of money creation. Section 3 calibrates the model to quantify the theory. Section 4 concludes.

2 Model

The model constructed here extends the standard monetary search model (Lagos and Wright, 2005) by introducing fractional reserve banking and unsecured credit. Time is discrete and there are two sub-periods for each time period: (1) a frictionless centralized market (CM, hereafter), where agents work, consume and adjust their balances, following after; (2) a decentralized market (DM, hereafter) where buyers and sellers meet and trade bilaterally. The DM trade features imperfect record-keeping and limited commitment. Due to these two frictions, some means of payment are needed. Below I describe the economic agents that live in this economy and the different types of DM meetings.

Buyers and Sellers The economy consists of a unit mass of buyers and a unit mass of sellers who discount their utility each period with discount factor β. The preferences of buyers and sellers for each period are:

$$\mathcal{U}^b = U(X) - H + u(q) \quad \text{and} \quad \mathcal{U}^s = U(X) - H - c(q)$$

where X is the CM consumption, H is the CM disutility from production and q is the DM consumption. As standard, assume U', u', $c' > 0$, U'', $u'' < 0$, $c'' \geq 0$ and $u(0) = c(0) = 0$. Consumption goods are perishable. One unit of H produces one unit of X in the CM. The efficient consumption in the CM and DM are denoted by X^* and q^*, which solve $U'(X^*) = 1$ and $u'(q^*) = c'(q^*)$, respectively.

The Bank A representative bank is born in the CM and die in the next CM. In the CM, the bank accepts deposit and decides how much to deposit at the central bank as reserves and how much banknote to issue. Assume that managing one unit of real deposit costs k units of CM good. The reserve earns a nominal interest rate of i_r. The bank extends loans by issuing banknotes. The loans are paid back with interest rate i_ℓ. Enforcing repayment is costly. The cost function is described by $\eta(\ell)$ and satisfies $\eta', \eta'' > 0$, where ℓ is the loan in real terms. The bank's lending is constrained by the reserve requirement, i.e., the bank cannot lend more than $(1-\chi)r/\chi$, where χ is the reserve requirement and r is the real balance of reserves.

Types of DM meetings There are three types of DM meeting. In DM1, there is no record-keeping device, and the seller can only recognize cash. In DM2, the seller can recognize cash, the claims on bank accounts and private banknotes. So, she accepts cash, deposit receipt and banknote. In DM3, in addition to the means of payment accepted in DM2, the buyer can trade using unsecured credit with credit limit $\bar{\delta}$ as the trading are monitored though imperfectly. The probability of joining a type j meeting is σ_j. The agents get to know which type of meeting they will be going in the preceding CM.[2]

The Central Bank The central bank controls the base money supply M in the CM. Let μ denote the base money growth rate. Then the changes in the real balance of base money can be written as

$$\mu M = M^{+} - M$$

where x^{+} is the value of (any variable) x in the next period. The base money is held in two ways: (1) C as currency in circulation i.e., outside money held by agents; (2) R as reserves held by a representative bank.

$$M = C + R$$

The central bank can control the base money supply in two ways. First, a central bank can conduct a lump-sum transfer or collect a lump-sum tax in the CM. Second, the central bank can increase the money supply by paying interest on reserves, i_r. Let T represents a lump-sum transfer (or tax if it is negative). The central bank's constraint is

$$\mu \phi M = \phi(M^{+} - M) = T + i_r \phi R$$

where ϕ is the price of money in terms of the CM consumption good.

2.1 The CM Problem

Buyers' Decisions At the beginning of the CM, each buyer's subsequent DM meeting type is realized. Therefore, the buyers' CM problem depends on their DM meeting type. Let $W_j^B(m, d, b, \ell, \delta)$ denote the CM value function where j is the type of the following DM meeting, m is the real cash holding, d is the real deposit balance, b is the real private banknote holding, ℓ is the real debt borrowed from the bank during the last CM period, and δ is the unsecured debt owed to the seller from the previous DM. All the state variables are expressed in unit of the current CM consumption good. Let τ denote the lump-sum transfer (or tax if it is negative) to the buyer in the CM. Now,

[2] This can be modeled as endogenous similar to Lester et al. (2012) or Lotz and Zhang (2016) but here we assume the types of meetings are exogenously given. Lester et al. (2012) endogenize the meeting types by allowing sellers' costly *ex ante* choice to acquire the technology for recognizing certain type of assets. Similarly, Lotz and Zhang (2016) studies the environment with costly record-keeping technology where sellers must invest in a record-keeping technology to accept credit.

consider the value of the CM. For an agent whose subsequent DM meeting type is realized as a j type DM meeting, the CM problem is

$$\begin{aligned} W_j^B(m,d,b,\ell,\delta) = &\max_{X,H,\hat{m}_j,\hat{d}_j,\hat{b}_j,\hat{\ell}_j} U(X) - H + \beta V_j^B(\hat{m}_j,\hat{d}_j,\hat{b}_j,\hat{\ell}_j) \\ \text{s.t. } (1+\pi)\hat{m}_j + (1+\pi)\hat{d}_j + X &= m + (1+i_d)d + b - (1+i_\ell)\ell - \delta + H + \tau \\ \hat{b}_j &= \hat{\ell}_j, \end{aligned} \tag{1}$$

where $\hat{m}_j$, $\hat{d}_j$, $\hat{b}_j$ and $\hat{\ell}_j$ are the real cash holding, real deposit balance, and real private banknote balance, and real debt balance, respectively, carried to the next DM. The FOCs are $U'(X) = 1$ and

$$-(1+\pi) + \beta\partial V_j^B(\hat{m}_j,\hat{d}_j,\hat{b}_j,\hat{\ell}_j)/\partial\hat{m}_j \leq 0, = \text{ if } \hat{m}_j > 0 \tag{2}$$

$$-(1+\pi) + (1+i_d)\beta\partial V_j^B(\hat{m}_j,\hat{d}_j,\hat{b}_j,\hat{\ell}_j)/\partial\hat{d}_j \leq 0, = \text{ if } \hat{d}_j > 0 \tag{3}$$

$$\beta\partial V_j^B(\hat{m}_j,\hat{d}_j,\hat{b}_j,\hat{\ell}_j)/\partial\hat{b}_j + \beta\partial V_j^B(\hat{m}_j,\hat{d}_j,\hat{b}_j,\hat{\ell}_j)/\partial\hat{\ell}_j \leq 0, = \text{ if } \hat{\ell}_j > 0. \tag{4}$$

The first term on the LHS of equation (2) is the marginal cost of acquiring cash. The second term is the discounted marginal value of carrying cash to the following DM. Therefore, the choice of $\hat{m}_j > 0$ equates the marginal cost and the marginal return on cash. Similar interpretation applies to equation (3) for the decision on deposit. For equation (4), the first term on the LHS captures the discounted marginal value of carrying privately issued banknotes from the CM to the following DM, and the second term on the LHS captures the discounted marginal cost of getting a bank loan. The envelope conditions for $W_j^B(m,d,b,\ell,\delta)$ are

$$\frac{\partial W_j^B}{\partial m} = 1, \quad \frac{\partial W_j^B}{\partial d} = 1 + i_d, \quad \frac{\partial W_j^B}{\partial b} = 1, \quad \frac{\partial W_j^B}{\partial \ell} = -(1+i_\ell), \quad \frac{\partial W_j^B}{\partial \delta} = 1,$$

for all $j = 1, 2, 3$, which implies $W_j^B(m,d,b,\ell,\delta)$ is linear. This linearity allows us to write

$$W_j^B(m,d,b,\ell,\delta) = m + (1+i_d)d + b - \delta - (1+i_\ell)\ell + W_j^B(0,0,0,0,0).$$

Let $W^B(m,d,b,\ell,\delta)$ be the buyers' expected value function before the CM at period t opens, i.e., before their subsequent DM meeting type is realized. Then we can write the buyer's expected value function in the CM as $W^B(m,d,b,\ell,\delta) = \sum \sigma_j W_j^B(m,d,b,\ell,\delta)$.

Sellers' Decisions A seller enters the CM with cash, m, deposits, d, private banknotes, b, and unsecured credit δ that a buyer owes to the seller from previous DM.

Let $W_j^S(m,d,b,0,\delta)$ be the sellers' value function in the CM at period t. Then this can be written as follows:

$$
\begin{aligned}
W_j^S(m,d,b,0,\delta) &= \max_{X,H,\hat{m}_j,\hat{d}_j,\hat{b}_j,\hat{\ell}_j} U(X) - H + \beta V_j^S(\hat{m}_j,\hat{d}_j,\hat{b}_j,\hat{\ell}_j) \\
\text{s.t. } (1+\pi)\hat{m}_j + (1+\pi)\hat{d}_j + X &= m + (1+i_d)d + b + \delta + H + \tau \\
\hat{b}_j &= \hat{\ell}_j
\end{aligned}
\tag{5}
$$

As we will see below, the DM terms of trade does not depend on the seller's portfolio, there is no incentive for sellers to carry any liquidity to the DM in the next period where the cost of holding liquidity is positive. Notice that $\hat{m}_j = \hat{d}_j = \hat{\ell}_j = \hat{b}_j = 0$ for the seller. Since the seller does not consume any goods in the DM, she has no incentive to bring cash, deposit balance and banknote to the next period DM because $\pi > \beta - 1$, $i > i_d$ and $i_\ell > 0$. In the equilibrium, one can check this holds. The envelope conditions are

$$
\frac{\partial W_j^S}{\partial m} = 1, \quad \frac{\partial W_j^S}{\partial d} = 1 + i_d, \quad \frac{\partial W_j^S}{\partial b} = 1, \quad \frac{\partial W_j^B}{\partial \delta} = 1,
$$

for all $j \in \{1,2,3\}$, which implies $W_j^S(m,d,b,0,\delta)$ is linear. This linearity also gives us following expression:

$$
W_j^S(m,d,b,0,\delta) = m + (1+i_d) + b + \delta + W_j^S(0,0,0,0,0).
$$

2.2 The DM Problem

In the DM, the buyer and seller trade bilaterally. Let q_j and p_j be the DM consumption and payment in type-j DM meeting then the bilateral trade is characterized by (p_j, q_j). This trade is subject to $p_j \leq z_j$ where z_j is the liquidity position of the buyer with a type-j meeting. The liquidity position for each type of buyer is given as:

$$
z_1 = m_1 \tag{6}
$$

$$
z_2 = m_2 + d_2(1+i_d) + b_2 \tag{7}
$$

$$
z_3 = m_3 + d_3(1+i_d) + b_3 + \bar{\delta}. \tag{8}
$$

To determine (p,q), i.e., to determine the terms of trade, Kalai (1977)'s proportional bargaining is used. The proportional bargaining solution is given by solving the following problem

$$
\max u(q) - p \quad s.t \quad u(q) - p = \theta\,[u(q) - c(q)]
$$

where $\theta \in [0,1]$ denotes the buyers' bargaining power. The amount of payment, p, is a function of DM consumption, q. This can be expressed as $p = v(q)$. For given bargaining protocol, the amount of trade is $v(q) = (1-\theta)u(q) + \theta c(q)$. Now, define *liquidity premium*, $\lambda(q)$, as follows:

$$\lambda(q) = \frac{u'(q)}{v'(q)} - 1 = \frac{\theta[u'(q) - c'(q)]}{(1-\theta)u'(q) + \theta c'(q)}$$

where $\lambda(q) > 0$ for $q < q^*$ and $\lambda(q^*) = 0$ with $\lambda'(q) < 0$ for $q \in [0, q^*)$. When $z_j \geq p^*$, the buyer has sufficient liquidity to purchase efficient DM output q^*. In this case, the payment to the seller will be $p^* = v(q^*)$.

Using the linearity of W_j^B, we can write a DM value function for a buyer in a type-j meeting as follows:

$$V_j^B(m_j, d_j, b_j, \ell_j) = u(q_j) - p_j + W^B(m_j, d_j, b_j, \ell_j, 0) \tag{9}$$

where $p_j \leq z_j$. The third term on the RHS is the continuation value when there is no trade. The rest of the RHS is the surplus from the DM trade. DM payments are constrained by $p_j \leq z_j$. With $v(q_j) = p_j$, differentiating V_j^B and substituting its derivatives into the FOCs from the CM problem yields

$$-(1+\pi) + \beta\big[1 + \lambda(q_1)\big] = 0 \tag{10}$$

$$-(1+\pi) + (1+i_d)\beta \leq 0, = \text{ if } d_1 > 0 \tag{11}$$

$$-i_\ell \leq 0, = \text{ if } \ell_1 > 0 \tag{12}$$

$$-(1+\pi) + \beta\big[1 + \lambda(q_i)\big] \leq 0, = \text{ if } m_j > 0 \text{ for } j = 2,3 \tag{13}$$

$$-(1+\pi) + (1+i_d)\beta\big[1 + \lambda(q_i)\big] \leq 0, = \text{ if } d_j > 0 \text{ for } j = 2,3 \tag{14}$$

$$-i_\ell + \lambda(q_i) \leq 0, = \text{ if } l_j > 0 \text{ for } j = 2,3, \tag{15}$$

where $q_j = \min\{q^*, v^{-1}(z_j)\}$ and $\lambda(q^*) = 0$.

Given the linearity of W_j^S, the sellers' DM value function is

$$V_j^S(m_j, d_j, b_j, \ell_j) = p_j - c(q_j) + W_j^S(m_j, d_j, b_j, \ell_j, 0).$$

2.3 The Bank's Problem

A representative bank maximizes its profit subject to its lending constraint where the enforcement cost function for a loan is given by $\eta(\ell) = \nu \ell^{\alpha}$ with $\alpha > 1$.

$$\max_{r,\ell} \quad i_r r + (-i_d - k)d + i_\ell \ell - \nu \ell^{\alpha} \tag{16}$$

$$s.t. \; \ell \leq \bar{\ell} = \frac{1-\chi}{\chi} r \tag{17}$$

$$r \leq d \tag{18}$$

Let λ_L denote the Lagrange multiplier for the lending constraint. Since it is straightforward to show $r = d$, the FOCs for the bank's problem can be written as

$$0 = i_r - i_d - k + \lambda_L \left(\frac{1-\chi}{\chi} \right) \tag{19}$$

$$0 = i_\ell - \alpha \nu \ell^{\alpha-1} - \lambda_L \tag{20}$$

There are two cases. For the first case, the bank's lending constraint is binding, i.e., $\lambda_L > 0$. In the second case, the bank's lending constraint is not binding, i.e., $\lambda_L = 0$. We call the first case "scarce reserves case," and the second "ample reserves case."

The Scarce-Reserves Case When constraint (17) is tight, $\lambda_L > 0$. Since the bank does not have enough reserves, it needs to acquire reserves to make more loans, which implies a binding constraint. With $\lambda_L > 0$, the bank's FOCs (19) and (20) give

$$0 = i_r - i_d - k + \left[i_\ell - \alpha \nu \left(\frac{1-\chi}{\chi} r \right)^{\alpha-1} \right] \frac{1-\chi}{\chi}. \tag{21}$$

The Ample-Reserves Case When constraint (17) is loose, $\lambda_L = 0$. Since the bank already has sufficient reserves, its lending constraint does not bind. Then the two FOCs for the bank's problem are:

$$0 = i_r - i_d - k \tag{22}$$

$$0 = i_\ell - \alpha \nu \ell^{\alpha-1}. \tag{23}$$

The bank's unconstrained optimal lending can be written as:

$$\ell^* = \left(\frac{i_\ell}{\alpha \nu} \right)^{\frac{1}{\alpha-1}}, \quad \ell^* < \bar{\ell} = \frac{1-\chi}{\chi} r.$$

2.4 Stationary Equilibrium

In this section, a monetary equilibrium is characterized. As a baseline model, we consider $\sigma_3 = 0$ ($\sigma_2 = 1 - \sigma_1$). An equilibrium with unsecured credit, ($\sigma_3 > 0$), will be introduced in the next section. I focus on a symmetric stationary monetary equilibrium where the same type of agents make the same decisions and real balances are constant over time. Since $\phi/\phi^+ = M^+/M = C^+/C = 1+\mu$, the net inflation rate, π, is equal to the currency growth rate, μ, in the stationary monetary equilibrium. From the Fisher equation, we have $1 + i = (1 + \mu)/\beta$. The market clearing conditions are

$$\sigma_2 \ell_2 = \ell \tag{24}$$

$$\sigma_2 d_2 = r = \phi R \tag{25}$$

$$\sigma_1 m_1 + \sigma_2 m_2 = \phi C, \tag{26}$$

where $M = C + R$. Note that $i \geq 0$, i.e., $\mu \geq \beta - 1$ is necessary for the existence of equilibrium.[3] Then we have the following definitions:

Definition 1 (**Stationary Equilibrium**). *Given monetary policy, (i, i_r, χ), a stationary monetary equilibrium consists of real balances $(m_j, d_j, \ell_j)_{j=1}^2$, allocations (q_1, q_2, X), and prices (i_ℓ, i_d), such that:*

(i) (i_d, i_ℓ, q_1, q_2) solves (19)-(20) and (10)-(15) with $q_1 = v^{-1}(z_1)$, $q_2 = v^{-1}(z_2)$ where $z_1 = m_1$ and $z_2 = m_2 + (1 + i_d)d_2 + \ell_2$.

(ii) The bank lending constraint satisfies, $\ell = \min(\bar{\ell}, \ell^*)$ where $\bar{\ell} = (1-\chi)r/\chi$ and $\ell^* = (i_\ell/\alpha\nu)^{\frac{1}{\alpha-1}}$.

(iii) Asset markets clear (24)-(26).

Given Definition 1, there are three types of equilibria, which are defined as follows:

Definition 2. *In an ample-reserve equilibrium, $\bar{\ell} > \ell^* \geq 0$. In a scarce-reserves equilibrium, $\ell^* \geq \bar{\ell} > 0$. In a no-banking equilibrium, $\bar{\ell} = r = 0$.*

For given definitions, the following results are proved in Appendix A where $L(\cdot) \equiv \lambda \circ v^{-1}(\cdot)$ for compact notation. Figure 3 illustrates those results.

[3] While the lower bound of the nominal interest rate is zero in this setting, one can relax this by introducing liquid assets or threats of theft. See Rocheteau, Wright and Xiao (2018b), Lee (2016) and Williamson et al. (2019) for detail.

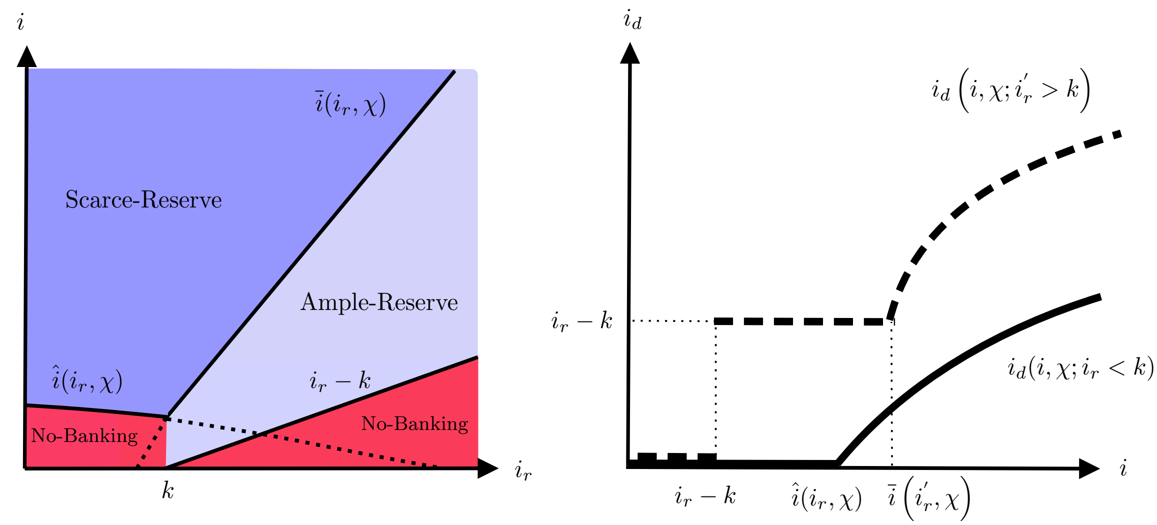

Figure 3: Equilibria and deposit rates

Proposition 1. *For a given equilibrium, the effects of policies are shown in Table 1.*

Table 1: Changes in policy

	$\ell^* \geq \bar{\ell} > 0$		$\bar{\ell} > \ell^* \geq 0$		$\bar{\ell} = 0$		
ζ	$\frac{\partial r}{\partial \zeta}$	$\frac{\partial i_d}{\partial \zeta}$	$\frac{\partial r}{\partial \zeta}$	$\frac{\partial i_d}{\partial \zeta}$	$\frac{\partial r}{\partial \zeta}$	$\frac{\partial i_d}{\partial \zeta}$	$\frac{\partial \ell^*}{\partial \zeta}$
i	-	+	-	0	0	0	+
i_r	+	+	+	+	0	0	-

Proposition 2. *Given (i_r, χ): (i) $\exists!$ ample-reserves equilibrium iff $i \in (0, \bar{i})$ and $i_r \geq k$; (ii) $\exists!$ scarce-reserves equilibrium iff $i \geq \max\{\hat{i}, \bar{i}\}$; (iii) $\exists!$ no banking equilibrium either $i \in [0, \hat{i})$ where $i_r < k$, or $i \in [0, i_r - k)$; and the threshold $\bar{i}$ satisfies*

$$\left(\frac{\frac{1+\bar{i}}{1+i_r-k} - 1}{\alpha\nu}\right)^{\frac{1}{\alpha-1}} = \sigma_2 L^{-1}\left(\frac{\bar{i} - i_r + k}{1 + i_r - k}\right)\frac{1-\chi}{1+\chi(i_r - k)}$$

and $\hat{i}$ satisfies

$$\frac{\chi}{1-\chi}[k - i_r] = \hat{i} - \alpha\nu[(1-\chi)\sigma_2 L^{-1}(\hat{i})]^{\alpha-1}.$$

Corollary 1. *The threshold $\bar{i}$ is increasing in i_r, and $\hat{i}$ is decreasing in i_r.*

Consider the case where the bank holds reserves. There is a downward-sloping demand for reserves both in the scarce-reserves equilibrium and the ample reserves equilibrium. The reserve requirement and the reserve balances determine the lending

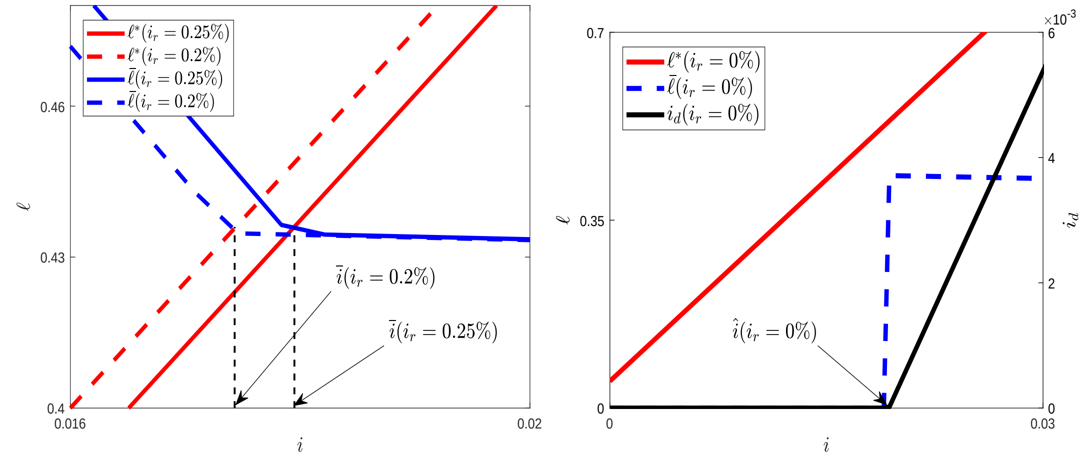

Figure 4: Determination of equilibrium type: ample-reserves vs. scarce-reserves (left) and no-banking vs. scarce-reserves (right)

limit. Given the reserve requirement, the lending limit is downward-sloping with respect to the nominal interest rate. On the contrary, due to monotone pass-through from the nominal interest rate to the bank's lending rate, the bank's unconstrained optimal lending is increasing in the nominal interest rate. Then, there exists a threshold of the nominal interest rate below which the lending limit is lower than the bank's unconstrained lending (scarce-reserves), and above which the lending limit is higher than the bank's unconstrained lending (ample-reserves). In other words, there is a critical value $\bar{i}$ that satisfies $\ell^* = \bar{\ell} = (1-\chi)r/\chi$.

However, when the equilibrium deposit rate is zero, agents have no incentive to deposit their balance into the bank and the bank does not hold reserves (no-banking). As shown in the Proposition 1, the deposit rate is monotone in the nominal interest rate, the deposit rate could be zero given some nominal interest rate. There exists the threshold $\hat{i}$, below which the deposit rate is zero and above which the deposit rate is positive.

Figure 4 shows some examples.[4] The left panel depicts $\ell^*(i)$ and $\bar{\ell}(i)$ for different interests on reserves. When the nominal interest rate is high enough ($i > \bar{i}$), we have a scarce-reserves equilibrium. Under the moderate nominal interest rate ($i < \bar{i}$), we have an ample-reserves equilibrium as long as deposit rates are positive. The right panel of Figure 4 shows an example that deposit rate can be zero. It depicts $\ell^*(i)$, $\bar{\ell}(i)$ (left axis) and $i_d(i)$ (right axis). In this example, when the nominal interest rate is high ($i > \hat{i}$), we have a scarce-reserves equilibrium. However, with the low nominal interest rate ($i < \hat{i}$), we have a no-banking equilibrium.

[4]These example use $u(q) = \sqrt{q}$, $c(q) = q$, $U(X) = \log(X)$, $\theta = 1$, $\nu = 0.02$, $k = 0.002$, $\alpha = 2$, $\sigma_1 = 0.1$ and $\chi = 0.5$.

The Ample-Reserves Equilibrium When $i \in (0, \bar{i})$ and $i_r \geq k$, there are sufficient reserves in the economy to lend ℓ^*. Thus, the unconstrained optimal lending is less than the lending limit. Since $\lambda(q_2) = i_\ell$ and $\lambda(q_2) = (i - i_d)/(1 + i_d)$, we have $i_\ell = (i - i_d)/(1 + i_d)$. Then we can formulate the demand for (supply of) loans from the private agents' (bank's) problem:

$$\text{Demand for loans:} \quad i_\ell = \lambda \circ v^{-1}[d_2(1 + i_r - k) + \ell_2] \tag{27}$$

$$\text{Supply of loans:} \quad i_\ell = \alpha\nu(\sigma_2\ell_2)^{\alpha-1} \tag{28}$$

Recall the bank's FOCs yield $i_d = i_r - k$ and $i_\ell = (i - i_r + k)/(1 + i_r - k)$ with $\lambda_L = 0$. Given (i, i_r), the equilibrium can be expressed by the following three equations:

$$\frac{1+i}{1+i_r-k} = 1 + \lambda(q_2) = 1 + \lambda \circ v^{-1}[d_2(1 + i_r) + \ell_2] \tag{29}$$

$$\sigma_2\ell_2 = \ell^* = \left(\frac{\frac{1+i}{1+i_r-k} - 1}{\alpha\nu}\right)^{\frac{1}{\alpha-1}} \tag{30}$$

$$i = \lambda(q_1) = \lambda \circ v^{-1}(m_1). \tag{31}$$

Given parameters and two policy variables (i, i_r), if $i \geq i_r - k$, the above three equations give a unique allocation $(q_1, q_2, m_1, d_2, \ell_2)$. In this type of equilibrium, $q_2 = q^*$ if $i = i_r - k$. Therefore, the DM2 meeting consumption can be efficient even though the economy is not under the Friedman rule. This result can be formally summarized in the following proposition.

Proposition 3. *In the ample-reserves case, DM2 consumption is efficient if* $i = i_r - k$.

The intuition behind the efficient DM2 consumption is straightforward. In many monetary models, a high inflation or a high interest rate increases the opportunity cost of holding money. In the environment where money is valued as a medium of exchange, having less liquidity in the economy because of an opportunity cost of holding money is inefficient. However, the interest on reserves provides a proportional return on reserves. If this return is properly distributed across agents, it eliminates the inefficiency which arises from the opportunity cost of holding money, which results in efficient DM2 consumption.

The Scarce-Reserves Equilibrium In the scarce-reserves case, the lending limit is less than the bank's optimal lending. Therefore, the lending constraint (17) binds, i.e., $\ell^* > \ell = \bar{\ell} = (1 - \chi)r/\chi$ where r represents the equilibrium reserves. In the binding

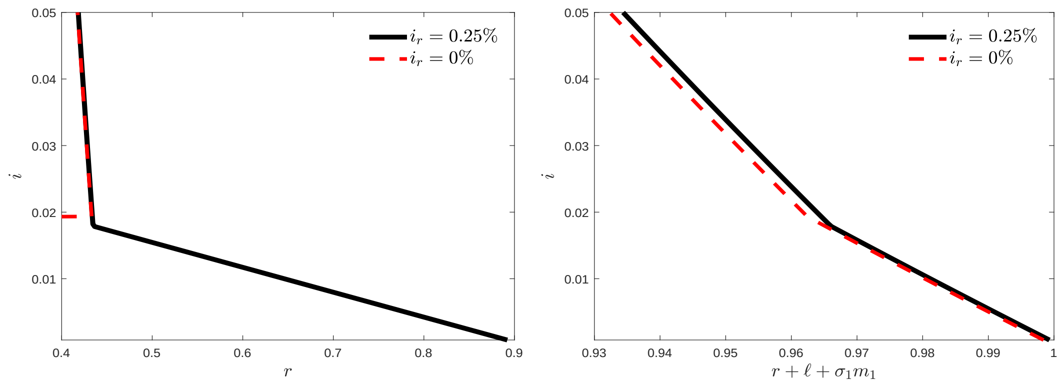

Figure 5: Demand for reserves and the monetary aggregate

case, the equilibrium lending is pinned down by r. To examine how r is determined, recall the equation (21) from the bank's problem:

$$0 = i_r - i_d - k + \left[i_\ell - \alpha\nu \left(\frac{1-\chi}{\chi} r \right)^{\alpha-1} \right] \frac{1-\chi}{\chi}$$

Since $\lambda(q_2) = i_\ell$ and $\lambda(q_2) = (i - i_d)/(1 + i_d)$, we have $i_\ell = (i - i_d)/(1 + i_d)$. Then, with the market clearing condition $r = \sigma_2 d$, the scarce-reserves equilibrium can be expressed by the following three equations, which gives the unique allocation $(q_1, q_2, m_1, d_2, \ell_2)$:

$$\text{Demand for reserves:} \quad k + i_d - i_r = \left[\frac{1+i}{1+i_d} - 1 - \alpha\nu \left(\frac{1-\chi}{\chi} r \right)^{\alpha-1} \right] \frac{1-\chi}{\chi} \tag{32}$$

$$\text{Supply of reserves:} \quad \frac{1+i}{1+i_d} = 1 + \lambda \circ v^{-1} \left[\frac{r}{\sigma_2} \left(\frac{1}{\chi} + i_d \right) \right] \tag{33}$$

$$i = \lambda(q_1) = \lambda \circ v^{-1}(m_1). \tag{34}$$

The No-Banking Equilibrium In the no-banking case, the deposit interest rate is negative or zero, $i_d \leq 0$. Since the return to holding deposits is dominated or equal to the return to holding currency, agents do not have any incentive to deposit their balances. With zero reserve, the lending limit is zero. Therefore, in this equilibrium, agents only use cash for DM trading. All agents hold the same balance of cash and consume the same amount of consumption goods.

$$i = \lambda(q_j) = \lambda \circ v^{-1}(m_j) \text{ for } j = 1, 2 \tag{35}$$

In this equilibrium, it is straightforward to see that DM consumptions are efficient when $i = 0$, i.e. the Friedman rule applies.

Regardless of the equilibrium type, there exists a downward-sloping demand for total liquidity. To illustrate this, I define the monetary aggregate as a sum of cash holdings, reserves, and banknotes in the economy. The right panel of Figure 5 shows that there exists a stable downward-sloping demand for monetary aggregates that shifts to the right as the central bank increases the interest on reserves. An increase in i_r shifts the money demand to the right, both in the scarce-reserves equilibrium and in the ample-reserves equilibrium. In the scarce-reserves case, higher i_r raises r and ℓ with a more loose lending constraint, and shifts the money demand to the right. However, in the ample-reserves equilibrium, higher i_r raises reserves r but it decreases ℓ. Lastly, a rise in i_r increases $\bar{i}$, which allows the monetary authority to induce the ample-reserves equilibrium with higher nominal interest rates. Figure 5 shows some examples.[5] The left panel depicts the demand for reserves for different interest rates on reserves. When $i_r = 0.25\%$, the reserves increase as i decreases, and this downward-sloping demand for reserves becomes much more elastic when $i < \bar{i}$, i.e., the ample-reserves case. So, we can observe a drastic increase in reserves under the ample-reserves equilibrium. When $i_r = 0\%$, while there is downward demand for reserves, the reserve balances are satiated at zero when $i < \hat{i}$, i.e., the no-banking equilibrium.

2.5 Stationary Equilibrium with Unsecured Credit

This section introduces the DM3 meeting with $\sigma_3 = 1 - \sigma_1 - \sigma_2 > 0$, so that the model allows use of unsecured credit as a means of payment. Consider the following market clearing conditions:

$$\sigma_2 \ell_2 + \sigma_3 \ell_3 = \ell \tag{36}$$

$$\sigma_2 d_2 + \sigma_3 d_3 = r = \phi R \tag{37}$$

$$\sigma_1 m_1 + \sigma_2 m_2 + \sigma_3 m_3 = \phi C, \tag{38}$$

where $M = C + R$. Then, define a stationary equilibrium with credit as follows.

Definition 3 (Stationary Equilibrium with Unsecured Credit). *Given monetary policy, i,i_r,χ and credit limit $\bar{\delta}$, a stationary monetary equilibrium consists of real balances $(m_j, d_j, \ell_j)_{j=1}^3$, allocations (q_1, q_2, q_3, X), and prices (i_ℓ, i_d), such that:*

(i) $(i_d, i_\ell, q_1, q_2, q_3)$ solves (19)-(20) and (10)-(15) with $q_1 = v^{-1}(z_1)$, $q_2 = \max\{q_1, v^{-1}(z_2)\}$ and $q_3 = \min\{q^*, \tilde{q}_3\}$ where $\tilde{q}_3 = \max\{q_2, v^{-1}(z_3)\}$, $z_1 = m_1$, $z_2 = m_2 + (1+i_d)d_2 + \ell_2$, and $z_3 = m_3 + (1 + i_d)d_3 + \ell_3 + \bar{\delta}$.

[5] The examples use the same functional forms and parameters in the Figure 4.

(ii) *The bank lending's constraint satisfies* $\ell = \min(\bar{\ell}, \ell^*)$, *where* $\bar{\ell} = (1-\chi)r/\chi$ *and* $\ell^* = (i_\ell/\alpha\nu)^{\frac{1}{\alpha-1}}$

(iii) *Asset markets clear* (36)-(38).

As defined in Definition 2, there are three types of equilibria: scarce-reserves equilibrium ($\bar{\ell} \leq \ell^*$); ample-reserves equilibrium ($\bar{\ell} > \ell^*$); and no-banking equilibrium ($\bar{\ell} = 0$). The following results are proved in Appendix A:

Proposition 4. *Let* $\tilde{\delta} \equiv L^{-1}\left(\frac{i-i_r+k}{1+i_r-k}\right)$ *and* $\hat{\delta} \equiv L^{-1}\left(\frac{i-\bar{i}_d}{1+\bar{i}_d}\right)$ *where* $\bar{i}_d$ *solves (32) and (33). For a given equilibrium and credit limit, we have comparative statics in Table 2.*

Table 2: Comparative statics

(a) Changes in policy

	$\ell^* \geq \bar{\ell} > 0$		$\bar{\ell} > \ell^* \geq 0$		$\bar{\ell} = 0$		
ζ	$\frac{\partial r}{\partial \zeta}$	$\frac{\partial i_d}{\partial \zeta}$	$\frac{\partial r}{\partial \zeta}$	$\frac{\partial i_d}{\partial \zeta}$	$\frac{\partial r}{\partial \zeta}$	$\frac{\partial i_d}{\partial \zeta}$	$\frac{\partial \ell^*}{\partial \zeta}$
i	-	+	-	0	0	0	+
i_r	+	+	+	+	0	0	-

(b) Changes in credit limit

	$\bar{\delta} < \hat{\delta}$	$\bar{\delta} > \hat{\delta}$	$\bar{\delta} < \tilde{\delta}$	$\bar{\delta} > \tilde{\delta}$	
	$\ell^* \geq \bar{\ell} > 0$		$\bar{\ell} > \ell^* \geq 0$		$\bar{\ell} = 0$
$\partial r/\partial \delta$	-	0	-	0	0
$\partial i_d/\partial \bar{\delta}$	+	0	0	0	0
$\partial \ell^*/\partial \bar{\delta}$	0	0	0	0	0

Proposition 5. *For given* $(i_r, \chi, \bar{\delta})$*: (i)* $\exists!$ *ample-reserves equilibrium iff* $i \in (0, \bar{\imath})$ *and* $i_r \geq k$*; (ii)* $\exists!$ *scarce-reserves equilibrium iff* $i \geq \max\{\hat{\imath}, \bar{\imath}\}$*; (iii)* $\exists!$ *no banking equilibrium either* $i \in [0, \hat{\imath})$ *where* $i_r < k$*, or* $i \in [0, i_r - k)$*; the threshold* $\bar{\imath}$ *satisfies*

$$\left(\frac{\frac{1+\bar{\imath}}{1+i_r-k} - 1}{\alpha\nu}\right)^{\frac{1}{\alpha-1}} = \frac{1-\sigma_1}{\chi} L^{-1}\left(\frac{\bar{\imath} - i_r + k}{1 + i_r - k}\right) \frac{1-\chi}{1+\chi(i_r-k)} - \frac{\sigma_3(1-\chi)\bar{\delta}}{1+\chi(i_r-k)}$$

if $\bar{\delta} < \tilde{\delta}$*, and* $\bar{\imath} = \bar{\bar{\imath}}$ *if* $\bar{\delta} \geq \tilde{\delta}$*; and* $\hat{\imath}$ *satisfies*

$$\frac{\chi}{1-\chi}[k - i_r] = \hat{\imath} - \alpha\nu\Big[(1-\chi)\big\{(1-\sigma_1)L^{-1}(\hat{\imath}) - \sigma_3\bar{\delta}\big\}\Big]^{\alpha-1}$$

if $\bar{\delta} < \hat{\delta}$*, and* $\hat{\imath} = \hat{\bar{\imath}}$ *if* $\bar{\delta} \geq \tilde{\delta}$.

Corollary 2. *The threshold* $\bar{\imath}$ *is increasing in* i_r*, and* $\hat{\imath}$ *is decreasing in* i_r*.*

Even after we introduce the unsecured credit, we still have similar results. Similar to Section 2.4, in the scarce-reserves equilibrium, lowering i increases r and decreases i_d . Thus, there is a threshold $\bar{\imath}$ that satisfies $\bar{\ell} = \ell^*$. Also, since i_d decreases as i decreases, there is a threshold $\hat{\imath}$ that satisfies $i_d = 0$. The critical point $\bar{\imath}$ makes the scarce-reserves case identical to the ample reserves case, while $\hat{\imath}$ is the critical point

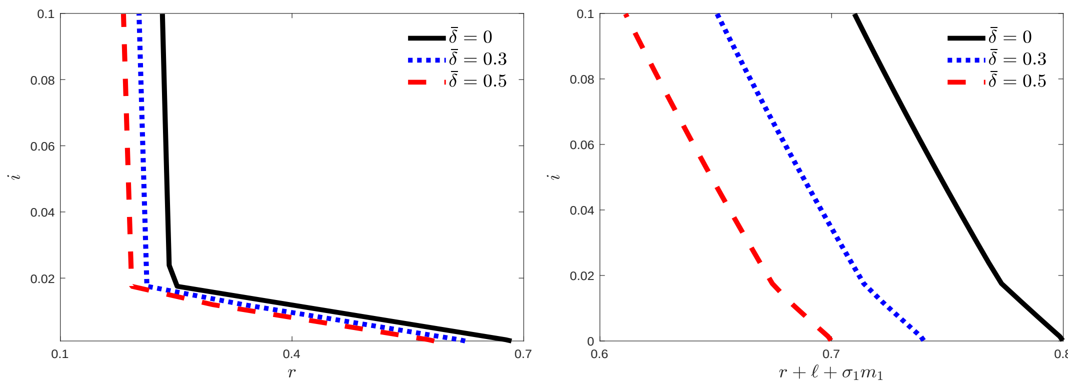

Figure 6: Demand for reserves and the monetary aggregate with different credit limits

that make the scarce-reserves case identical to the no-banking case. We can characterize the types of equilibria for given policy variables and parameters. Each type of equilibrium and its properties are discussed below.

The Ample-Reserves Equilibrium For $\bar{\delta} \geq L^{-1}\left(\frac{i-i_r+k}{1+i_r-k}\right)$, we have $q_3 = \min\{v^{-1}(\bar{\delta}), q^*\}$, $d_3 = \ell_3 = 0$ and (r, m_1, i_d) solves equations (29)-(31) with $r = \sigma_2 d_2$. For $\bar{\delta} < L^{-1}\left(\frac{i-i_r+k}{1+i_r-k}\right)$, an ample-reserves equilibrium with unsecured credit consists of the following equations where $(i_r, i, \chi, \bar{\delta})$ is given:

$$\frac{1+i}{1+i_r-k} = 1 + \lambda \circ v^{-1}(d_2(1+i_r) + \ell_2)$$

$$\frac{1+i}{1+i_r-k} = 1 + \lambda \circ v^{-1}(\bar{\delta} + d_3(1+i_r) + \ell_3)$$

$$\sigma_2 \ell_2 + \sigma_3 \ell_3 = \ell^* = \left(\frac{\frac{1+i}{1+i_r-k} - 1}{\alpha \nu}\right)^{\frac{1}{\alpha-1}}$$

$$i = \lambda \circ v^{-1}(m_1)$$

with $r = \sigma_2 d_2 + \sigma_3 d_3$. An ample-reserves equilibrium with unsecured credit is also similar to the ample-reserves equilibrium from Section 2.4, except for two minor differences: (i) $r = \sigma_2 d_2 + \sigma_3 d_3$, (ii) $\bar{\delta}$ crowds out d_3 and ℓ_3. While $\bar{\delta}$ crowds out d_3 and ℓ_3 as substitutes, $d_2(1+i_r)+\ell_2$ is equal to $\bar{\delta}+d_3(1+i_r)+\ell_3$ as long as $\bar{\delta} < L^{-1}\left(\frac{i-i_r+k}{1+i_r-k}\right)$.

The Scarce-Reserves Equilibrium For $\bar{\delta} \geq L^{-1}\left(\frac{i-\bar{i}_d}{1+\bar{i}_d}\right)$ where $\bar{i}_d$ solves equation (32) and (33), we have $q_3 = \min\{v^{-1}(\bar{\delta}), q^*\}$, $d_3 = \ell_3 = 0$ and (r, m_1, i_d) solves equation (32)-(34). When $\bar{\delta} < L^{-1}\left(\frac{i-\bar{i}_d}{1+\bar{i}_d}\right)$, a scarce-reserves equilibrium with unsecured credit consists of the following equations with given $(i_r, i, \chi, \bar{\delta})$

$$i = \lambda \circ v^{-1}(m_1) \tag{39}$$

$$k + i_d - i_r = \left[\frac{1+i}{1+i_d} - 1 - \alpha\nu\left(\frac{1-\chi}{\chi}r\right)^{\alpha-1}\right]\left(\frac{1-\chi}{\chi}\right) \tag{40}$$

$$\frac{1+i}{1+i_d} = 1 + \lambda \circ v^{-1}\left[d_2\left(\frac{1}{\chi} + i_d\right)\right] \tag{41}$$

$$\frac{1+i}{1+i_d} = 1 + \lambda \circ v^{-1}\left[\bar{\delta} + d_3\left(\frac{1}{\chi} + i_d\right)\right] \tag{42}$$

where $r = \sigma_2 d_2 + \sigma_3 d_3$. The scarce-reserves equilibrium with unsecured credit is mostly the same to the scarce-reserves equilibrium from Section 2.4 except $r = \sigma_2 d_2 + \sigma_3 d_3$ and $\bar{\delta}$ crowds out d_3. As a result, $d_2\,(1/\chi + i_d)$ is equal to $\bar{\delta} + d_3\,(1/\chi + i_d)$.

The No-Banking Equilibrium When $\bar{\delta} \geq L^{-1}(i)$, $q_3 = \min\{v^{-1}(\bar{\delta}), q^*\}$, $m_3 = 0$ and $m_j = L^{-1}(i)$ for $j = 1, 2$. When $\bar{\delta} < L^{-1}(i)$, $m_3 = L^{-1}(i) - \bar{\delta}$ and $m_j = L^{-1}(i)$ for $j = 1, 2$. A no-banking equilibrium with unsecured credit is similar to the no-banking equilibrium from Section 2.4, except $\bar{\delta}$ crowds out m_3 which results in $m_1 = m_2 = \bar{\delta} + m_3$.

The Role of Access to Unsecured Credit We also can check the effect of changes in the access to unsecured credit, σ_3. An increase in σ_3 implies that more buyers can use unsecured credit. Some fraction of DM2 buyers are changed into DM3 buyers and they hold less amount of deposit balance compared to the amount they had.

Under the scarce-reserve equilibrium, an increase in σ_3 lowers r. While total real balance of reserves are decreased, both d_2 and d_3 increase as σ_3 increases due to the increases in i_d. An increase in σ_3 lowers r also under the ample-reserve equilibrium. The Appendix A verifies the following:

Proposition 6. *For a given $(i, i_r, \chi, \bar{\delta})$, we have comparative statics in Table 3*

Table 3: Credit access change

$\ell^* \geq \bar{\ell} > 0$		$\bar{\ell} > \ell^* \geq 0$		$\bar{\ell} = 0$		
$\frac{\partial r}{\partial \sigma_3}$	$\frac{\partial i_d}{\partial \sigma_3}$	$\frac{\partial r}{\partial \sigma_3}$	$\frac{\partial i_d}{\partial \sigma_3}$	$\frac{\partial r}{\partial \sigma_3}$	$\frac{\partial i_d}{\partial \sigma_3}$	$\frac{\partial \ell^*}{\partial \sigma_3}$
-	+	-	0	0	0	0

3 Quantitative Analysis

To evaluate the theory quantitatively, I calibrate the model to match several targets using pre-2008 data. Using calibrated parameters, I compare the model predictions with the data pre-2008 and post-2008 periods. Given the parameters, the stationary equilibrium is characterized by $(i, i_r, \chi, \bar{\delta})$. The required reserves ratio is computed by dividing the required reserves by total checkable deposits. While the first three series are easy to obtain, it is hard to get the unsecured credit limit, $\bar{\delta}$, either from macro and micro data. The unsecured credit limit is computed using the unsecured credit to output ratio. In the model, the unsecured credit to output ratio is given by $\sigma_3\bar{\delta}/(B + \sum_j^3 \sigma_j z_j)$, so we can compute $\bar{\delta}$ using the model with the given policies (i, i_r, χ) and parameters. Following Krueger and Perri (2006), the revolving consumer credit is used as the unsecured credit. For this exercise, I generate simulated data by using 4 series: (i) nominal interest rates[6]; (ii) the interest on reserves; (iii) the required reserve ratio; and (iv) the unsecured credit to GDP ratio.

3.1 Calibration

The utility functions for the DM and the CM are $u(q) = Aq^{1-\gamma}/(1-\gamma)$ and $U(X) = \log(X)$ implying $X^* = 1$ (a normalization). The cost function for the DM is $c(q) = q$. The fraction of buyers who can use unsecured credit is set as $\sigma_3 = 0.4783$.[7] I set $\alpha = 2$ for convexity of the enforcement cost.[8] When σ_1 is set, σ_2 will be set directly since $\sigma_2 = 1 - \sigma_1 - \sigma_3$. The remaining 6 parameters $(\theta, A, \gamma, k, \nu, \sigma_1)$ are set to match the following six targets: (i) the average retail market markup; (ii) the average credit share of the DM transactions, $\sigma_3\delta/DM$; (iii) the average currency to deposit ratio, C/D; (iv) the average reserves to output ratio, R/Y; (v) the average currency to output ratio, C/Y; (vi) the semi-elasticity of C/Y to i where i denotes the nominal interest rate. The targets are computed based on 1968-2007 data, except for the markup which uses the average from 1993 to 2007. The bargaining power θ is set to match the DM markup to the retail markup from the Annual Retail Trade Survey (ARTS) (see https://www.census.gov/programs-surveys/arts.html). The average ratio of gross margins to sales from 1993-2007 is 0.2776, implying the average markup is

[6] I use 3-month Treasury-bill rates as standard. Section 3.5 and Appendix B.3 check robustness by using different measures of monetary policy, which suggests that the main results are not overly sensitive to the choice for the measure of monetary policy.

[7] Durkin (2000) provides the percentage of households with a credit card from 1970 to 1998. (1970, 1977, 1983, 1989, 1995, 1998) The average of this period is 47.83%. Since this percentage during period varies over time, sensitivity analyses are included in Appendix B.3.

[8] Since there is not sufficient justifications for $\alpha = 2$ either from macro-level or from micro-level data, sensitivity analyses are included in Appendix B.3.

Table 4: Model parametrization

Parameter	Value	Target/source	Data	Model
External Parameters				
enforcement cost curvature, α	2	Set directly		
DM3 matching prob, σ_3	0.4783	Durkin (2000)		
Jointly Determined Parameters				
bargaining Power, θ	0.454	avg. retail markup	1.384	1.384
enforcement cost level, ν	0.020	avg. UC/DM	0.387	0.378
DM1 matching prob, σ_1	0.189	avg. C/D	0.529	0.564
deposit operating cost, k	0.002	avg. R/Y	0.016	0.016
DM utility level, A	0.618	avg. C/Y	0.044	0.044
DM utility curvature, γ	0.398	semi-elasticity of C/Y to i	-3.716	-3.724

Note: C, R, DM, UC, Y denote currency in circulation, reserves, DM transactions, unsecured credit and nominal GDP, respectively. D denotes inside money.

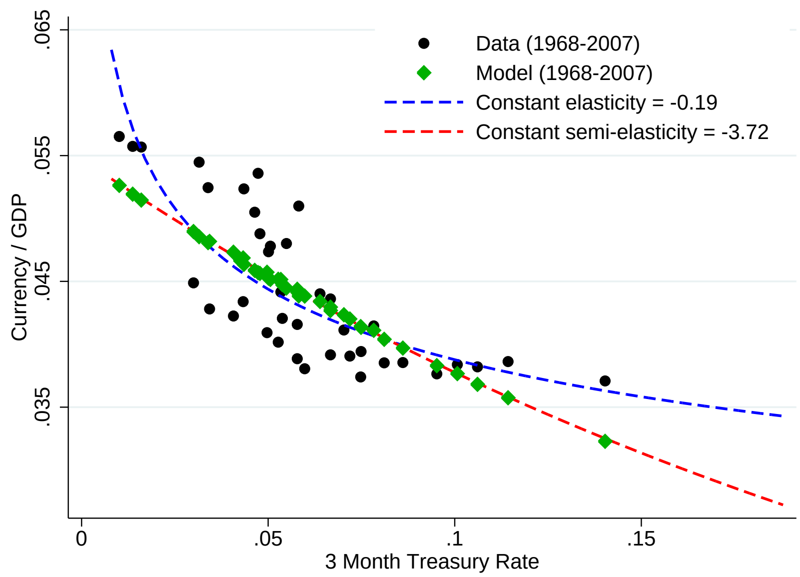

Figure 7: Money demand for currency

1.3844. Changing A shifts the currency demand curve to match the currency to output ratio, and setting γ matches the semi-elasticity of C/Y. The costs of operating deposit services and issuing loans from the bank, captured by k and ν, are set to match the reserves to output ratio and the unsecured credit to DM transaction ratio. Lastly, I set σ_1 to match the currency-deposit ratio. The calibrated parameters and the targets are summarized in Table 4, and the calibrated money demand of currency is shown in Figure 7.

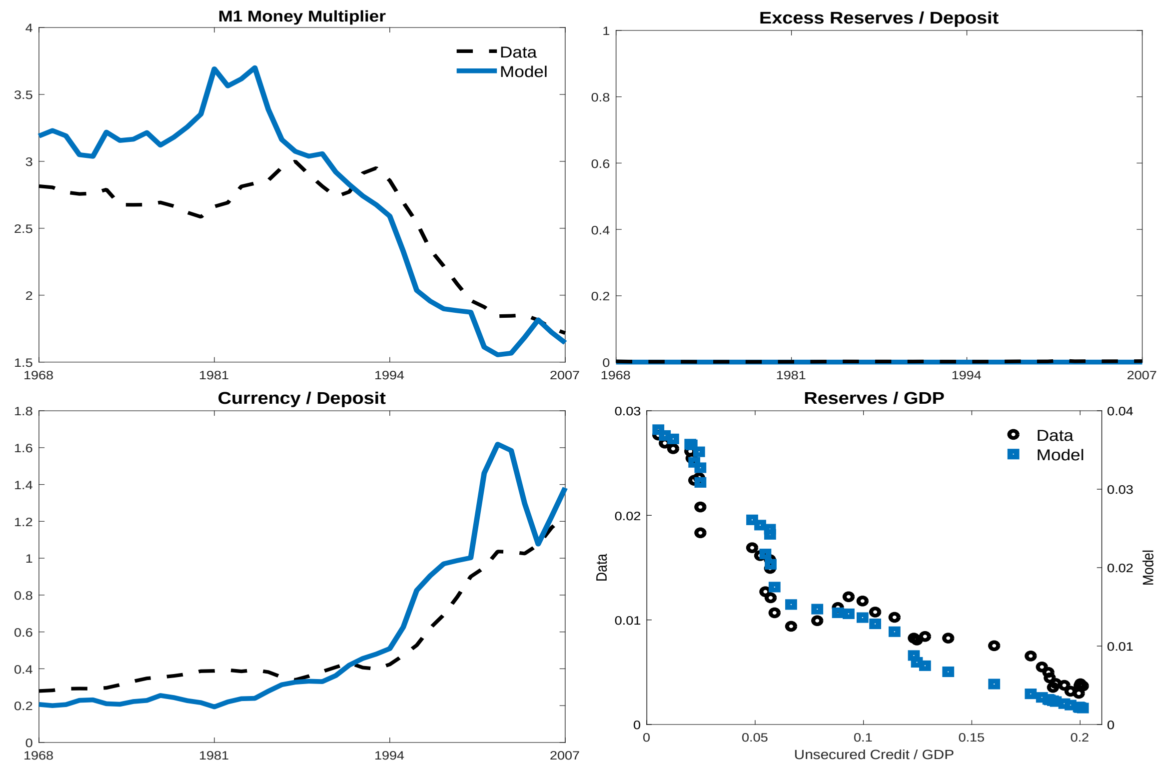

Figure 8: In-sample fit: 1968-2007

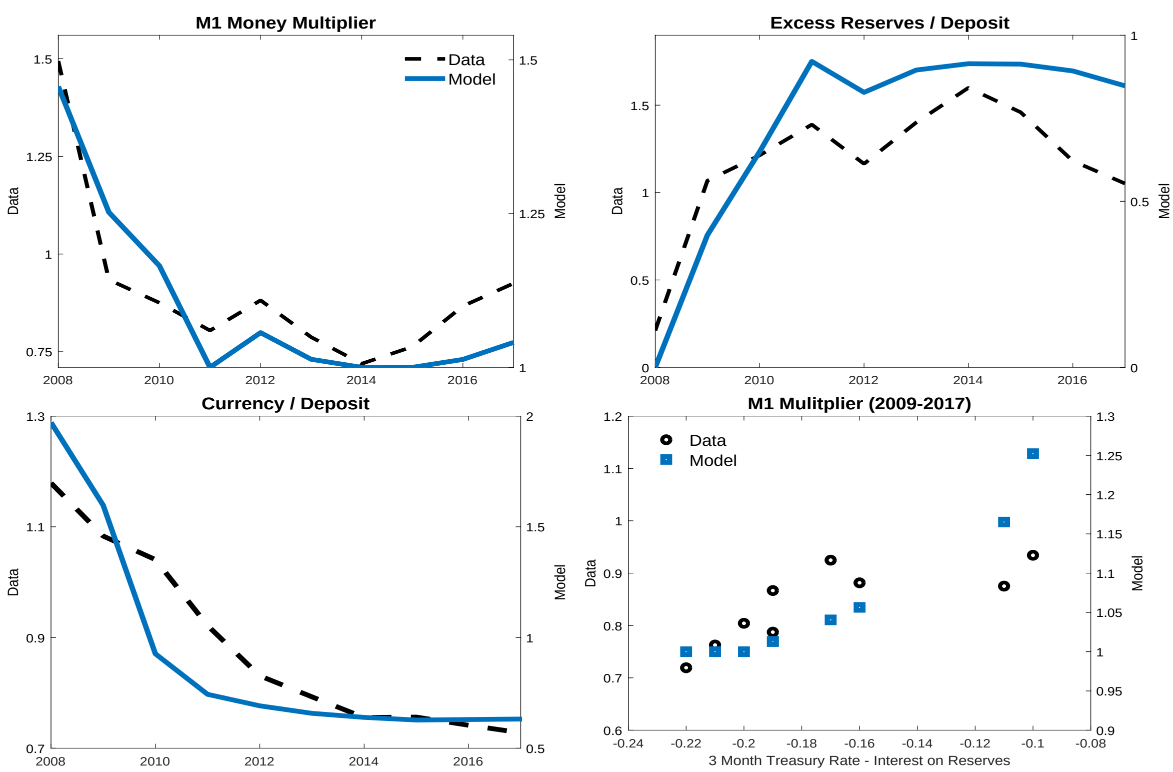

Figure 9: Out-of-sample fit: 2008-2018

3.2 Results and the Model Fit

Figure 8 compares the model and data for the sample period, 1968 -2007. The top-left panel of Figure 8 shows the M1 money multiplier from 1968 to 2007. The model generates decreases in the M1 multiplier during 1987-2007, while the peak was in 1987 in the data and 1984 in the model. During this period, the excess reserves to deposit ratio had been almost zero both in the data and in the model, suggesting the US economy had been in the scarce-reserves equilibrium. In the model, the declining trend of the M1 multiplier in the pre-2008 period is driven by the increase in unsecured credit, which crowds out the inside money (private banknotes, and reserves) but not currency. This induces increases in the currency to deposit ratio, as shown in the bottom-left panel of Figure 8. The bottom-right panel of Figure 8 compares how unsecured credit crowds out reserves in the model and the data.

The next step is to evaluate model projections by comparing them with the data after 2007. Overall, the model can match the patterns in the data. Timeplots of Figure 9 compares the model projections for the M1 money multiplier, the excess reserves to deposit ratio, and the currency to deposit ratio with data from 2008 to 2018. The model implied series show similar patterns to the actual data series. The model can generate the change in the equilibrium type, from scarce-reserves to ample-reserves, and a similar pattern of excess reserves to deposit ratio. This change in the equilibrium type is represented by a huge drop in the money multiplier in the top-left panel and a huge increase in the excess reserves to deposit ratio in the top-right panel.

Regression estimates shown in Table 5 illustrate the main mechanism of the model. Columns (1) and (2) show the regression coefficient estimates using the following equation for 1968-2007.

$$\text{Reserves}_t/\text{GDP}_t = \beta_0 + \beta_1 \text{UnsecuredCredit}_t/\text{GDP}_t + \beta_2 \texttt{Tbill3}_t + \epsilon_t$$

Since all three series have a unit root and are cointegrated, both in the data and in the model-generated series, the coefficients are estimated using the canonical cointegrating regression proposed by Park (1992).[9] The estimated negative coefficient on the 3 month T-bill rate suggests a downward sloping demand for reserves with respect to the interest rate; but other coefficients on unsecured credit suggest that this demand for reserves can shift as the credit condition changes, as shown in Figure 6. This is consistent with Proposition 4, and the model-implied regression gives similar results.

Columns (3)-(4) regress the M1 multiplier on the 3 month T-bill rate and the interest on reserves and Columns (5)-(6) regress the excess reserves ratio on the same

[9]Unit root and cointegration test results are reported in Appendix B.

Table 5: Model-implied regression coefficients, model vs. data

Dependent Variable:	Reserves/GDP (1968-2007)		M1 Money Multiplier (2009-2017)		Excess Reserve/Deposit (2009-2017)	
	Data (1)	Model (2)	Data (3)	Model (4)	Data (5)	Model (6)
Unsecured Credit/GDP	-0.123^{***}	-0.190				
	(0.004)					
3 Month T-bill Rate	-0.083^{***}	-0.072	1.004^{***}	1.999	-2.447^{***}	-3.771
	(0.011)		(0.156)		(0.423)	
Interest on Reserves			-0.892^{***}	-2.034	2.137^{***}	3.842
			(0.150)		(0.405)	
R^2	0.876	0.849	0.652	0.922	0.612	0.855

Notes: Columns (1)-(2) report the canonical cointegrating regression (CCR) estimates. First stage long-run variance estimation for CCR is based on Bartlett kernel and lag 1. Columns (3)-(6) report OLS estimates. For (3) and (5) Newey-West standard errors with lag 1 are reported in parentheses. ***, **, and * denote significance at the 1, 5, and 10 percent levels, respectively. Intercepts are included but not reported.

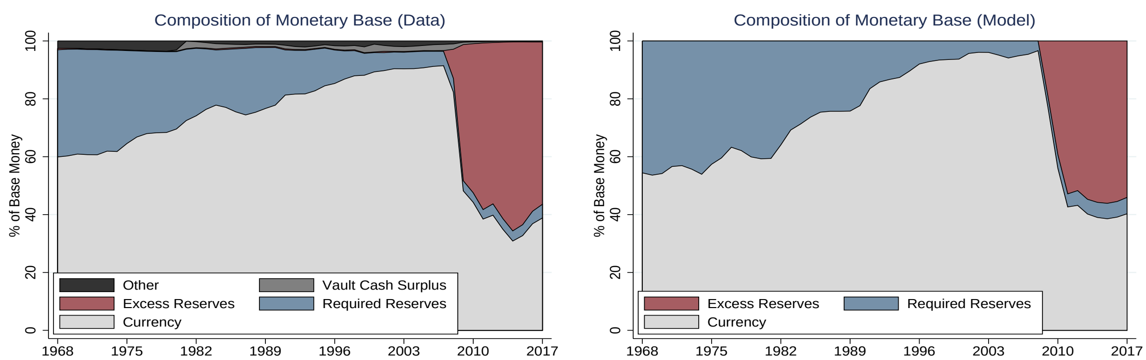

Figure 10: Composition of monetary base: data vs. model[10]

variables. Because the number of observations in the data are too small, Columns (3) and (5) use data from the 1st quarter of 2009 to the 4th quarter of 2017. Regressions using the data and the model-implied series provide similar results. Based on the regression results in (3)-(6), for a given interest on reserves, raising the 3 month T-bill rates increases the M1 multiplier while it decreases excess reserves. At the same time, for a given 3-month treasury rate, lowering the interest on reserves decreases the M1 multiplier while it increases excess reserves. In the model, when the bank faces higher interests on reserves the bank holds more reserves and does not lend as much as before. This is because interest on reserves yields profits to the bank with low cost but lending is associated with the enforcement cost. This increases the excess reserve ratio and lowers the money multiplier.

[10] "Other" denotes all the monetary base which are not included as currency component of M1, reserves, and vault cash surplus. Since monetary base is either currency or reserves in the model,

The model also provides the composition of the monetary base over time. Figure 10 compares the composition of the monetary base from the data and the model. The model successfully generates the changes for each component of the monetary base - currency, required reserves, and excess reserves - both before and after 2008.

A Digression on Model Fit For the post-2007 period, while the model projections can match the patterns in the data well, they do not fit very well in levels. This discrepancy is from the fact that the theoretical lower bound for the money multiplier is one in the model. In reality, however, the U.S. economy has experienced M1 multipliers lower than 1.[11] There are two potential explanations for this.

One possible reason is that monetary policy can be conducted in different ways compared to the lump-sum transfer in the model. In the model, all the base money is distributed to agents through the lump-sum transfer, and they keep some fraction in their bank accounts. Reserves are in the bank deposits in this setup, and this implies the money multiplier can not be lower than one. In contrast to most of the monetary models that assume money is injected as a lump-sum transfer across the agents (buyers and sellers, in this model), much money injection is made to the banking system directly in the real economy. For example, in the quantitative easing program, the Fed purchased large amounts of financial assets from financial intermediaries and gave them the same amount in reserves. These reserves are directly injected into the banking system and this is different from lump-sum transfers. In this case, reserves can only be held by banks, not by the public, and banks do not lend out reserves. One may need to consider a more explicit mechanism for monetary policy implementation.[12]

The other possible reason is that reserves could be kept in saving account or time deposits, which is in M2 but not in M1. Even though one assumes that monetary base is distributed through a lump-sum transfer, it does not have to be kept in checkable account. In this case, there is no discrepancy between the data and the theoretical lower bound for the money multiplier since the M2 money multiplier has never been lower than 1.[13] From a balance sheet point of view, reserves are recorded as cash asset on the commercial bank's balance sheet because reserves are held as an account for

right figure from the model does not have "Other" and "vault cash surplus".

[11]The M1 multiplier of the US was lower than one from December 2008 (0.975) until June 2018 (0.991).

[12]Previous works on the explicit model of the interbank market with monetary policy implementation include Armenter and Lester (2017), Afonso et al. (2019), Bianchi and Bigio (2014) and Chiu, Eisenschmidt and Monnet (2020). Those models explicitly describe search frictions and the market structure of the interbank market for reserves while this paper assumes a centralized market for reserves with a representative bank. Noting that the Fed controls the effective federal funds rates which are interbank rates, introducing the interbank market can allow more realistic monetary transmission.

[13]The lowest M2 money multiplier during 1959-2019 was 2.812 at Aug of 2014.

Table 6: Welfare cost

Interest on Reserves Reserve Requirement	$i_r = 0\%$ $\chi = 0.1$ (1)	$i_r = 0.25\%$ $\chi = 0.1$ (2)	$i_r = 0\%$ $\chi = 0.5$ (3)	$i_r = 0.25\%$ $\chi = 0.5$ (4)	$i_r = 0\%$ $\chi = 0.9$ (5)	$i_r = 0.25\%$ $\chi = 0.9$ (6)
q_1	0.141	0.141	0.141	0.141	0.141	0.141
$q_2 = q_3$	0.263	0.263	0.204	0.206	0.152	0.154
$1 - \Delta$	0.0167	0.0167	0.0331	0.0324	0.0655	0.0638

the commercial banks at the Federal Reserve Bank, but deposits are liabilities. In this case, reserves cannot exceed total liabilities (total deposits) but it can exceed checkable deposits.

3.3 Welfare

I measure the cost of inflation, Δ, by computing the percentage reduction of the DM surplus that agents would accept against their optimal DM consumption level, in other words, DM consumption under the Friedman rule.[14] The total DM surplus in the unit of the CM good is:

$$\Omega^{DM}(\mu, \chi, i_r) = \sum_{j=1}^{3} \sigma_j \{u[q_j(\mu, \chi, i_r)] - q_j(\mu, \chi, i_r)\}, \tag{43}$$

and the CM surplus for buyers and sellers is constant $U(X^*) - X^*$. Since DM consumption under the Friedman rule is q^*, Δ solves $\Omega^{DM}(\mu, \chi, i_r) = u(q^*\Delta) - q^*$. To focus on the effect of different instruments of monetary policy, I set $\bar{\delta} = 0$.

Table 6 shows the welfare cost under 10% inflation. Column (1) presents the results under the 10% reserve requirement and zero interest on reserves. Under the 10% reserve requirement, without interest on reserves, the welfare cost for the DM is 1.7%. With a 50% reserve requirement, we have a higher welfare loss at 3.3% as shown in Column (3). These losses are much less than the 6.8% from Lagos and Wright (2005). However, with the 90% reserve requirement, the welfare cost for the DM is 6.6% in Column (5), which is similar to previous work on the welfare cost of inflation. The difference comes from the fact that an increase of the real stock of inside money improves welfare. Higher reserve requirement can decrease welfare in the stationary equilibrium.

Paying interest on reserve can reduce welfare cost. If the reserve requirement is 90%, without interest on reserves, the welfare cost for the DM is 6.6%, and paying

[14] Following Lucas (2000) and Bethune, Choi and Wright (2020), I set $\beta = 0.9709$ which implies the inflation rate at the Friedman rule is $\mu^0 = -0.0291$ i.e., $i = 0$.

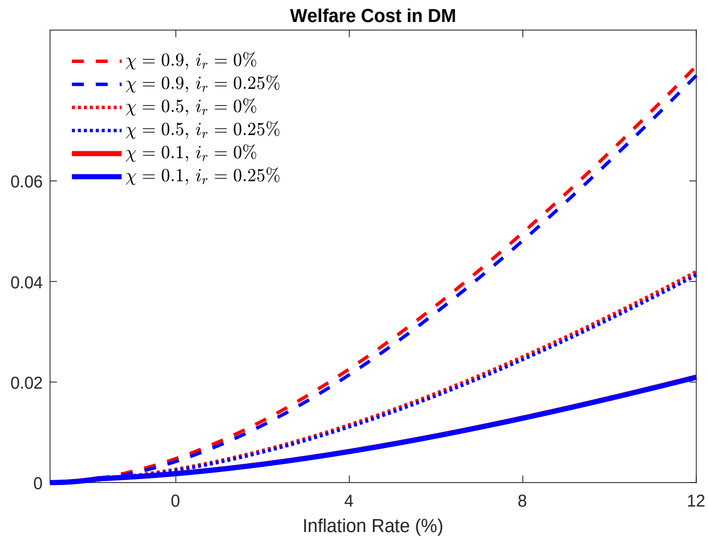

Figure 11: Cost of inflation

0.25% interest on reserves can reduce this to 6.4%. This difference becomes smaller if the reserve requirement is lower. For example, under a 10% reserve requirement, paying interest on reserves improves welfare far less.

Figure 11 illustrates the effects of different degrees of inflation ranging from 0 to 120% and how inflation can change depending on different reserve requirements and different interest rates on reserves. Each curve denotes the welfare cost of inflation for the different reserve requirements and interest rate on reserves. The welfare cost is monotonically increasing in inflation and each curve can be shifted down by paying interest on reserves or lowering the reserve requirement. The difference becomes smaller as the economy faces low inflation, and the Friedman rule gives zero welfare loss.

3.4 Counterfactual Analysis

In this section, I use calibrated parameters to assess how the money multiplier and currency deposit ratio would be changed by setting different reserve requirements. I also demonstrate that it is important to distinguish the effect of reserve requirement from that of credit.

The top panel of Figure 12 shows the counterfactual under different reserve requirement while keeping $(i, i_r, \bar{\delta})$ the same as in the benchmark case. With lower reserve requirement, the money multiplier increases. However, we see a trend of decreasing multiplier regardless of reserve requirement. As illustrated in Table 5, this gradual decrease in the money multiplier since the 1980s is driven by an increase in unsecured credit in the model. The currency deposit ratio increases with higher reserve

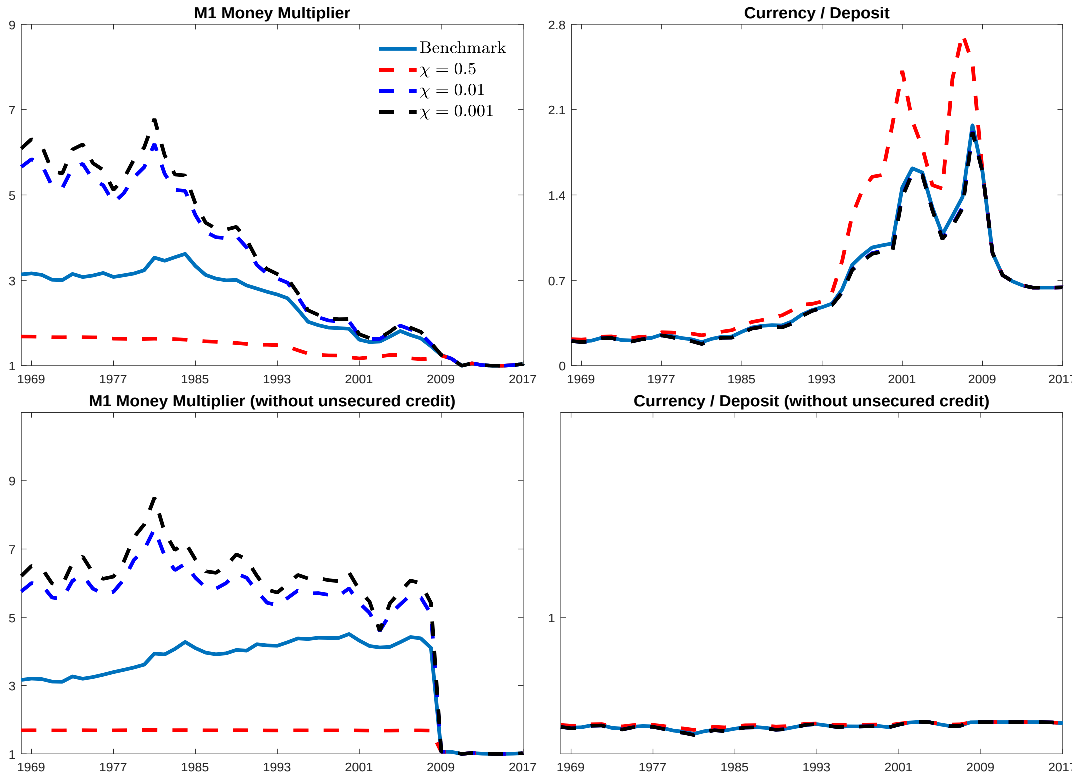

Figure 12: Counterfactual analysis

requirement and all the cases show the similar trend.

The bottom panel of Figure 12 shows the counterfactual under different reserve requirements and $\bar{\delta} = 0$ with (i, i_r) the same as in the data. In the bottom-right panel, we see almost no changes in the currency deposit ratio. Since the money demand for currency and inside money is stable, if unsecured credit did not crowd out inside money, there would not be substantial increase in the currency deposit ratio as the U.S. economy witnessed. In the case of the money multiplier, while it shows a stationary pattern around 6-7, it drops drastically after the Federal Reserve started paying interest on reserves and lowered the nominal interest rate. This suggests that the gradual decline of the money multiplier from the early 1980s to 2007 can be attributed to an increase in unsecured credit, while the dramatic decline in the money multiplier since 2008 can be explained by the monetary policies of the Federal Reserve.

3.5 Robustness

This section briefly summaries a few results from Appendix B.3 and B.4. In Appendix B.3, to check whether the enforcement cost curvature parameter α is sensitive, I recalibrate α as 2.2 or 1.8 instead of 2 which is the benchmark parameter. Also, to check the sensitivity of the credit trade fraction parameter, σ_3, I set this to 0.3 instead of 0.4783. Changing these parameters does not have a significant impact on the results. I also recalibrate the model using a different measure of monetary aggregate, the M2 instead of the M1. The model can match the patterns in the data. Appendix B.4 examines the sensitivity of the results using different measures of monetary policy target: the federal funds rate, the commercial paper rate, and the inflation rate. Using different measure does not change the main results except for the case using the inflation rate.

4 Concluding Remarks

This paper develops a monetary-search model with fractional banking and unsecured credit, and studies the money creation process. In the fractional reserve system, the money is created when the bank makes loans. The bank's lending, however, can be constrained by the reserve requirement and the reserves. In the model, there are three types of equilibrium: (i) the scarce-reserves equilibrium; (ii) the ample-reserves equilibrium; and (iii) the no-banking equilibrium.

In the no-banking equilibrium, all the base money is held by the buyers and there are no deposits in the economy. In the scarce-reserves equilibrium, the bank's lending is constrained by the reserve requirement and reserves. So, the money creation process depends on the reserves and the reserve requirement. In the ample-reserve equilibrium, the bank's lending is not constrained by the reserve requirement or reserves because of sufficient reserves. In this case, the bank's lending depends on the nominal interest rates and the interest on reserves. Regardless of the equilibrium types, there exists a money demand relationship, and the Friedman rule is optimal. In the ample-reserves equilibrium, there exists an optimal interest on reserves which makes the non-cash DM meeting consumption efficient without the Friedman rule.

There exists a threshold of the nominal interest rate between the scarce-reserves equilibrium and the ample-reserves equilibrium. If the nominal interest rate is lower than the threshold, the equilibrium is the ample-reserves equilibrium, while if the nominal interest rate is higher than the threshold, the equilibrium is the scarce-reserves equilibrium. In addition, this threshold is increasing in interest on reserves. Therefore, paying interest on reserves with low nominal interest rates can move the economy from scarce reserves equilibrium to the ample reserves equilibrium, which is consistent with what we have seen in the US economy. The quantitative analysis can generate simulated data that resemble the actual data. This paper provides evidence from the model and the data that suggests that the dramatic changes in the money multiplier after 2008 are mainly driven by the introduction of the interest on reserves with the low nominal interest rate.

The paper can be extended in various ways. While I focus on the centralized market for the reserves with a representative bank, in reality, the market for reserves is a decentralized interbank market and each bank has a heterogeneous balance sheet with a different portfolio. Therefore, one can further investigate how much the market structure and heterogeneity matter for the transmission of monetary policy (e.g., Afonso and Lagos, 2015; Armenter and Lester, 2017; Afonso, Armenter and Lester, 2019). Second, it would be worthwhile to study how inside creation via making loans is be related to investment and firms' dynamics (e.g., Ennis, 2018; Bianchi and Bigio, 2014; Altermatt, 2019). This will allow us to understand the investment channels of monetary policy more explicitly. Moreover, I assume that bank assets are composed of loans and reserves. But commercial banks' assets are mainly composed of securities, loans, and reserves. Extending the model to incorporate banks portfolio choice and analyzing the role of investment, financial regulation, and monetary policy can open up other research avenues (e.g., Rocheteau, Wright and Zhang, 2018a).

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Appendix

Appendix A Proofs

Proof of Proposition 1 and 2. First, consider the no-banking equilibrium.

$$i = L(m_j) \text{ for } j = 1, 2$$

with $r = 0$ and $i_d = 0$. In the no-banking case, it is straight forward to show

$$\frac{\partial r}{\partial i} = 0, \quad \frac{\partial i_d}{\partial i} = 0, \quad \frac{\partial r}{\partial i_r} = 0, \quad \frac{\partial i_d}{\partial i_r} = 0.$$

Now consider the ample-reserves case:

$$\frac{1+i}{1+i_r-k} = 1 + L\left[\frac{r}{\sigma_2}(1+i_r) + \ell_2\right], \quad \sigma_2\ell_2 = \ell^* = \left(\frac{\frac{1+i}{1+i_r-k} - 1}{\alpha\nu}\right)^{\frac{1}{\alpha-1}}.$$

Clearly, ℓ^* is increasing in i and decreasing in i_r. Since $\frac{r}{\sigma_2}(1+i_r) + \ell_2$ is decreasing in i, r is decreasing in i. Similarly, since $\frac{r}{\sigma_2}(1+i_r) + \ell_2$ is increasing in i_r, r is increasing in i_r. In the ample-reserves case, the deposit rate is determined as $i_d = i_r - k$, and it is increasing in i_r but $\partial i_d/\partial i = 0$.

Next, consider the scarce-reserves case with $\lambda_L > 0$. Letting r^d and r^s represent as demand for reserves and as supply for reserves, respectively, where $r^d = \sigma_2 r^s = r$ in the equilibrium and we have following supply and demand equations.

$$\underbrace{\frac{1+i}{1+i_d} - 1 - \alpha\nu\left(\frac{1-\chi}{\chi}r^d\right)^{\alpha-1} - (k + i_d - i_r)\left(\frac{\chi}{1-\chi}\right)}_{\equiv D} = 0, \tag{44}$$

$$\underbrace{\frac{1+i}{1+i_d} - 1 - L\left[r^s\left(\frac{1}{\chi} + i_d\right)\right]}_{\equiv S} = 0. \tag{45}$$

While the demand for reserves has downward slope with $\partial r^d/\partial i_d < 0$, the supply for reserves could exhibit backward bending supply. If $i_d > \hat{i}_d$ where $\hat{i}_d$ solves $(1+i)/(1+\hat{i}_d)^2 + L'(\cdot)r^s = 0$, there could be backward bending supply for reserve. In other words $(1+i)/(1+i_d)^2 + L'(\cdot)r^s < 0$ when $i_d > \hat{i}_d$ while $(1+i)/(1+i_d)^2 + L'(\cdot)r^s > 0$ when $i_d < \hat{i}_d$. However, one can check following conditions are held at the scarce-reserves equilibrium:

$$\sigma_2\chi L^{-1}(i) \leq r^d(0), \quad r^d(i) < \frac{\sigma_2\chi p^*}{\chi i + 1}. \tag{46}$$

This exclude the backward bending part of reserve supply since these condition implies equilibrium deposit rate i_d is always less than $\hat{i}_d$ at the equilibrium implying $(1 +$

$i)/(1+i_d)^2 + L'(\cdot)r^s > 0$. Then, solving equations for scarce-reserve case gives a unique solution because of the downward demand, upward supply and the conditions (46).

Given these results, using implicit function theorem we have

$$D_{i_d} di_d + D_\chi d\chi + D_r dr + D_i di + D_{i_r} di_r = 0 \quad (47)$$

$$S_{i_d} di_d + S_\chi d\chi + S_r dr + S_i di + S_{i_r} di_r = 0 \quad (48)$$

where

$$D_{i_d} = -\frac{1+i}{(1+i_d)^2} - \frac{\chi}{1-\chi}$$
$$D_\chi = \alpha(\alpha-1)\nu\left(\frac{1-\chi}{\chi}r\right)^{\alpha-2}\frac{r}{\chi^2} - (k+i_d-i_r)\frac{1-2\chi}{(1-\chi)^2}$$
$$D_{i_r} = \frac{\chi}{1-\chi}$$
$$D_r = -\alpha(\alpha-1)\nu\left(\frac{1-\chi}{\chi}r\right)^{\alpha-2}\frac{1-\chi}{\chi}$$
$$D_i = \frac{1}{1+i_d}$$

$$S_{i_d} = -\frac{1+i}{(1+i_d)^2} - \frac{r}{\sigma_2}L'\left[\frac{r}{\sigma_2}\left(\frac{1}{\chi}+i_d\right)\right]$$
$$S_\chi = L'\left[\frac{r}{\sigma_2}\left(\frac{1}{\chi}+i_d\right)\right]\frac{r}{\sigma_2}\frac{1}{\chi^2}$$
$$S_{i_r} = 0$$
$$S_r = -L'\left[\frac{r}{\sigma_2}\left(\frac{1}{\chi}+i_d\right)\right]\frac{1}{\sigma_2}\left(\frac{1}{\chi}+i_d\right)$$
$$S_i = \frac{1}{1+i_d}$$

Since $\lambda' < 0$ we have $L' < 0$. With $(1+i)/(1+i_d)^2 + L'(\cdot)r^s > 0$ we have following comparative statics results.

$$\frac{\partial r}{\partial i} = \frac{\begin{vmatrix} D_{i_d} & -D_i \\ S_{i_d} & -S_i \end{vmatrix}}{\begin{vmatrix} D_{i_d} & D_r \\ S_{i_d} & S_r \end{vmatrix}} < 0, \quad \frac{\partial r}{\partial i_r} = \frac{\begin{vmatrix} D_{i_d} & -D_{i_r} \\ S_{i_d} & -S_{i_r} \end{vmatrix}}{\begin{vmatrix} D_{i_d} & D_r \\ S_{i_d} & S_r \end{vmatrix}} > 0,$$

$$\frac{\partial i_d}{\partial i} = \frac{\begin{vmatrix} -D_i & D_r \\ -S_i & S_r \end{vmatrix}}{\begin{vmatrix} D_{i_d} & D_r \\ S_{i_d} & S_r \end{vmatrix}} > 0, \quad \frac{\partial i_d}{\partial i_r} = \frac{\begin{vmatrix} -D_{i_r} & D_r \\ -S_{i_r} & S_r \end{vmatrix}}{\begin{vmatrix} D_{i_d} & D_r \\ S_{i_d} & S_r \end{vmatrix}} > 0$$

For given (χ, i, i_r), $\bar{\ell}$ and r is decreasing in i and i_d is monotone and increasing function of i in scarce-reserves equilibrium.

Next step is finding the conditions for each case. Let $\bar{i}$ solves

$$\sigma_2 L^{-1}\left(\frac{\bar{i}-i_r+k}{1+i_r-k}\right)\frac{1-\chi}{1+\chi i_r} = \left(\frac{\frac{1+\bar{i}}{1+i_r-k}-1}{\alpha\nu}\right)^{\frac{1}{\alpha-1}}$$

and let $\hat{i}$ solves

$$\frac{\chi}{1-\chi}[k-i_r] = \hat{i} - \alpha\nu[(1-\chi)\sigma_2 L^{-1}(\hat{i})]^{\alpha-1}$$

$\bar{i}$ is a threshold that make scarce-reserves case and ample-reserves case equivalent and $\hat{i}$ is a threshold that make scarce-reserves case and no-banking case equivalent by making $i_d = 0$. Since ℓ^* is increasing in i and $\bar{\ell}$ is decreasing in i, if $\bar{i} > \hat{i}$ (i) The stationary monetary equilibrium is a scarce-reserve equilibrium when $i \geq \bar{i}$. (ii) The stationary monetary equilibrium is an ample-reserves equilibrium when $i < \bar{i}$ and $i_r >$

k. Recall the condition of no-banking equilibrium, $i_d \leq 0$. Then, it is straightforward to show following. If $\hat{i} > \bar{i}$, (i) The stationary monetary equilibrium is a scarce-reserve equilibrium when $i \geq \hat{i}$. (ii) The stationary monetary equilibrium is a no-banking equilibrium when $i < \hat{i}$.

Therefore we have comparative statics results in Table 1 and for given (i_r, χ): (i) $\exists!$ ample-reserves equilibrium iff $i \in (0, \bar{i})$ and $i_r \geq k$; (ii) $\exists!$ scarce-reserves equilibrium iff $i \geq \max\{\hat{i}, \bar{i}\}$; (iii) $\exists!$ no banking equilibrium either $i \in [0, \hat{i})$ where $i_r < k$, or $i \in [0, i_r - k)$.

■

Proof of Corollary 1. First, it is straight forward to show

$$\frac{\partial \hat{i}}{\partial i_r} = \frac{-\chi}{(1-\chi)\left\{1 - (1-\chi)\sigma_2 L'^{-1}(\hat{i})\alpha(\alpha-1)\nu[(1-\chi)\sigma_2 L^{-1}(\hat{i})]^{\alpha-2}\right\}} < 0$$

To show $\partial \bar{i}/\partial i_r > 0$, recall

$$\frac{1-\chi}{\chi} r = \left(\frac{\frac{1+\bar{i}}{1+i_r-k} - 1}{\alpha\nu}\right)^{\frac{1}{\alpha-1}}.$$

Since $\partial r/\partial i_r > 0$, both in the scarce-reserves case and ample reserve case, a rise in i_r increases the left-hand side. For given $\bar{i}$, a rise in i_r increases the left-hand side. To have equality, $\bar{i}$ need to be increased. Therefore, a rise in i_r increases $\bar{i}$. ■

Proof of Proposition 4 and 5. First, consider no-banking equilibrium:

$$i = L(m_j) \text{ for } j = 1, 2$$

and $m_3 = L^{-1}(i) - \bar{\delta}$ with $r = 0$ and $i_d = 0$. In the no-banking case, it is straight forward to show:

$$\frac{\partial r}{\partial i} = 0, \quad \frac{\partial i_d}{\partial i} = 0, \quad \frac{\partial r}{\partial i_r} = 0, \quad \frac{\partial i_d}{\partial i_r} = 0.$$

Now, consider the ample-reserves case:

$$\frac{1+i}{1+i_r-k} = 1 + L\left[d_2(1+i_r) + \ell_2\right] = 1 + L\left[d_3(1+i_r) + \ell_3 + \bar{\delta}\right]$$

$$\sigma_2\ell_2 + \sigma_3\ell_3 = \ell^* = \left(\frac{\frac{1+i}{1+i_r-k} - 1}{\alpha\nu}\right)^{\frac{1}{\alpha-1}}$$

where $r = \sigma_2 d_2 + \sigma_3 d_3$. Clearly, ℓ^* is increasing in i and decreasing in i_r. Since $d_2(1+i_r) + \ell_2$ and $d_3(1+i_r) + \ell_3 + \bar{\delta}$ are decreasing in i and increasing in i_r, r is decreasing in i and increasing in i_r. This implies $\bar{\ell}$ is decreasing in i and increasing in i_r. In the ample-reserves case, the deposit rate is determined as $i_d = i_r - k$, and it is increasing in i_r but $\partial i_d/\partial i = 0$. Next, let's consider the scarce-reserve equilibrium. In case for $\bar{\delta} \leq \hat{\delta}$, comparative statics are same as Section 2.4 since DM3 agent with sufficiently high credit limit does not hold any liquidity from CM to DM. Changes in $\bar{\delta}$ does not affect to other variables except q_3. Now focus on case with $\bar{\delta} < \hat{\delta}$. Recall equation (40)-(42):

$$\frac{1+i}{1+i_d} - 1 - \alpha\nu\left(\frac{1-\chi}{\chi}[\sigma_2 d_2 + \sigma_3 d_3]\right)^{\alpha-1} - (k + i_d - i_r)\left(\frac{\chi}{1-\chi}\right) = 0, \tag{49}$$

$$\frac{1+i}{1+i_d} - 1 - L\left[\bar{\delta} + d_3\left(\frac{1}{\chi} + i_d\right)\right] = 0, \tag{50}$$

$$\frac{1+i}{1+i_d} - 1 - L\left[d_2\left(\frac{1}{\chi} + i_d\right)\right] = 0. \tag{51}$$

Similar with the proofs for the Proposition 1 and 2, we can establish the uniqueness of this system of equations with $(1+i)/(1+i_d)^2 + L'(\cdot)d_j > 0$ for $j = 1, 2$. We can rewrite above equations as below:

$$\underbrace{\frac{1+i}{1+i_d} - 1 - \alpha\nu\left(\frac{1-\chi}{\chi}\left[(\sigma_2+\sigma_3)d_3 + \frac{\sigma_2\chi\bar{\delta}}{1+\chi i_d}\right]\right)^{\alpha-1} - (k + i_d - i_r)\left(\frac{\chi}{1-\chi}\right)}_{\equiv\Omega} = 0, \tag{52}$$

$$\underbrace{\frac{1+i}{1+i_d} - 1 - L\left[\bar{\delta} + d_3\left(\frac{1}{\chi} + i_d\right)\right]}_{\equiv\Lambda} = 0. \tag{53}$$

where $d_2 = d_3 + \frac{\chi\bar{\delta}}{1+\chi i_d}$. Differentiating above two equation gives

$$\Omega_{i_d} di_d + \Omega_\chi d\chi + \Omega_d dd + \Omega_i di + \Omega_{i_r} di_r = 0 \tag{54}$$

$$\Lambda_{i_d} di_d + \Lambda_\chi d\chi + \Lambda_d dd + \Lambda_i di + \Lambda_{i_r} di_r = 0. \tag{55}$$

where

$$\begin{aligned}
&\Omega_{i_d} = -\tfrac{1+i}{(1+i_d)^2} - \tfrac{\chi}{1-\chi} && \Lambda_{i_d} = -\tfrac{1+i}{(1+i_d)^2} - d_3 L'\left[\bar{\delta} + d_3\left(\tfrac{1}{\chi} + i_d\right)\right] \\
&\Omega_\chi = \alpha(\alpha-1)\nu\left(\tfrac{1-\chi}{\chi}d\right)^{\alpha-2}\tfrac{d}{\chi^2} - (k+i_d-i_r)\tfrac{1-2\chi}{(1-\chi)^2} && \Lambda_\chi = L'\left[\bar{\delta} + d_3\left(\tfrac{1}{\chi} + i_d\right)\right] d_3 \tfrac{1}{\chi^2} \\
&\Omega_{i_r} = \tfrac{\chi}{1-\chi} && \Lambda_{i_r} = 0 \\
&\Omega_{d_3} = -\sigma_3\alpha(\alpha-1)\nu\left(\tfrac{1-\chi}{\chi}d\right)^{\alpha-2}\tfrac{1-\chi}{\chi} && \Lambda_{d_3} = -L'\left[\bar{\delta} + d_3\left(\tfrac{1}{\chi} + i_d\right)\right](\tfrac{1}{\chi} + i_d) \\
&\Omega_i = \tfrac{1}{1+i_d} && \Lambda_i = \tfrac{1}{1+i_d} \\
&\Omega_{\bar{\delta}} = -\alpha(\alpha-1)\nu\left(\tfrac{1-\chi}{\chi}d\right)^{\alpha-2}\tfrac{\sigma_2\chi}{1+\chi i_d} && \Lambda_{\bar{\delta}} = -L'\left[\bar{\delta} + d_3\left(\tfrac{1}{\chi} + i_d\right)\right]
\end{aligned}$$

Since $\lambda' < 0$ we have $L' < 0$. With $(1+i)/(1+i_d)^2 + L'(\cdot)d^j > 0$ for $j = 1, 2$, we have following comparative statics results:

$$\frac{\partial d_3}{\partial i}=\frac{\begin{vmatrix}\Omega_{i_d} & -\Omega_i\\ \Lambda_{i_d} & -\Lambda_i\end{vmatrix}}{\begin{vmatrix}\Omega_{i_d} & \Omega_{d_3}\\ \Lambda_{i_d} & \Lambda_{d_3}\end{vmatrix}}<0,\quad \frac{\partial d_3}{\partial i_r}=\frac{\begin{vmatrix}\Omega_{i_d} & -\Omega_{i_r}\\ \Lambda_{i_d} & -\Lambda_{i_r}\end{vmatrix}}{\begin{vmatrix}\Omega_{i_d} & \Omega_{d_3}\\ \Lambda_{i_d} & \Lambda_{d_3}\end{vmatrix}}>0,\quad \frac{\partial d_3}{\partial \bar{\delta}}=\frac{\begin{vmatrix}\Omega_{i_d} & -\Omega_{\bar{\delta}}\\ \Lambda_{i_d} & -\Lambda_{\bar{\delta}}\end{vmatrix}}{\begin{vmatrix}\Omega_{i_d} & \Omega_{d_3}\\ \Lambda_{i_d} & \Lambda_{d_3}\end{vmatrix}}<0$$

$$\frac{\partial i_d}{\partial i}=\frac{\begin{vmatrix}-\Omega_i & \Omega_{d_3}\\ -\Lambda_i & \Lambda_{d_3}\end{vmatrix}}{\begin{vmatrix}\Omega_{i_d} & \Omega_{d_3}\\ \Lambda_{i_d} & \Lambda_{d_3}\end{vmatrix}}>0,\quad \frac{\partial i_d}{\partial i_r}=\frac{\begin{vmatrix}-\Omega_{i_r} & \Omega_{d_3}\\ -\Lambda_{i_r} & \Lambda_{d_3}\end{vmatrix}}{\begin{vmatrix}\Omega_{i_d} & \Omega_{d_3}\\ \Lambda_{i_d} & \Lambda_{d_3}\end{vmatrix}}>0,,\quad \frac{\partial i_d}{\partial \bar{\delta}}=\frac{\begin{vmatrix}-\Omega_{\bar{\delta}} & \Omega_{d_3}\\ -\Lambda_{\bar{\delta}} & \Lambda_{d_3}\end{vmatrix}}{\begin{vmatrix}\Omega_{i_d} & \Omega_{d_3}\\ \Lambda_{i_d} & \Lambda_{d_3}\end{vmatrix}}>0$$

with $\frac{\partial d_2}{\partial i}=d_3\frac{\partial d_3}{\partial i}-\frac{\chi\bar{\delta}}{(1+\chi i_d)^2}\frac{\partial i_d}{\partial i}<0$ and $\frac{\partial d_2}{\partial i_r}=-\frac{\frac{1+i}{(1+i_d)^2}+d_2L'}{(\frac{1}{\chi}+i_d)L'}\frac{\partial i_d}{\partial i_r}>0$.

Next step is finding the conditions for each case. Let $\bar{\imath}$ solves

$$\left(\frac{\frac{1+\bar{\imath}}{1+i_r-k}-1}{\alpha\nu}\right)^{\frac{1}{\alpha-1}}=\frac{1-\sigma_1}{\chi}L^{-1}\left(\frac{\bar{\imath}-i_r+k}{1+i_r-k}\right)\frac{1-\chi}{1+\chi(i_r-k)}-\frac{\sigma_3(1-\chi)\bar{\delta}}{1+\chi(i_r-k)}$$

and let $\hat{\imath}$ solves

$$\frac{\chi}{1-\chi}[k-i_r]=\hat{\imath}-\alpha\nu\Big[(1-\chi)\big\{(1-\sigma_1)L^{-1}(\hat{\imath})-\sigma_3\bar{\delta}\big\}\Big]^{\alpha-1}.$$

$\bar{\imath}$ is a threshold that make scarce-reserves case and ample-reserves case equivalent and $\hat{\imath}$ is a threshold that make scarce-reserves case and no-banking case equivalent by making $i_d=0$. Since ℓ^* is increasing in i and $\bar{\ell}$ is decreasing in i, if $\bar{\imath}>\hat{\imath}$ (i) The stationary monetary equilibrium is a scarce-reserve equilibrium when $i\geq\bar{\imath}$. (ii) The stationary monetary equilibrium is an ample-reserves equilibrium when $i<\bar{\imath}$ and $i_r>k$. Recall the condition of no-banking equilibrium is $i_d\leq 0$. Then it is straightforward to show following. If $\hat{\imath}>\bar{\imath}$, (i) The stationary monetary equilibrium is a scarce-reserve equilibrium when $i\geq\hat{\imath}$. (ii) The stationary monetary equilibrium is a no-banking equilibrium when $i<\hat{\imath}$.

Therefore we have comparative statics results in Table 2 and for given (i_r,χ): (i) $\exists!$ ample-reserves equilibrium iff $i\in(0,\bar{\imath})$ and $i_r\geq k$; (ii) $\exists!$ scarce-reserves equilibrium iff $i\geq\max\{\hat{\imath},\bar{\imath}\}$; (iii) $\exists!$ no banking equilibrium either $i\in[0,\hat{\imath})$ where $i_r<k$, or $i\in[0,i_r-k)$. ■

Proof of Corollary 2. First, it is straight forward to show

$$\frac{\partial\hat{\imath}}{\partial i_r}=\frac{-\chi}{(1-\chi)\left\{1-(1-\chi)(1-\sigma_1)L'^{-1}(\hat{\imath})\alpha(\alpha-1)\nu[(1-\chi)(1-\sigma_1)L^{-1}(\hat{\imath})-\sigma_3\bar{\delta}]^{\alpha-2}\right\}}<0$$

To show $\partial\bar{\iota}/\partial i_r > 0$, recall

$$r\frac{1-\chi}{\chi} = \left(\frac{\frac{1+\bar{\iota}}{1+i_r-k} - 1}{\alpha\nu}\right)^{\frac{1}{\alpha-1}}.$$

Since $\partial r/\partial i_r > 0$ both in the scarce-reserves case and ample reserve case, a rise in i_r increases the left-hand side. For given $\bar{\iota}$, a rise in i_r increases the left-hand side. To have equality, $\bar{\iota}$ need to be increased. Therefore, a rise in i_r increases $\bar{\iota}$. ■

Proof of Proposition 6. Similar to the proof of Proposition 4, substituting and total differentiating (40)-(42) gives

$$\begin{bmatrix} \Omega_{i_d} & \Omega_{d_3} \\ \Lambda_{i_d} & \Lambda_{d_3} \end{bmatrix} \begin{bmatrix} di_d/d\sigma_3 \\ dd_3/d\sigma_3 \end{bmatrix} = \begin{bmatrix} -\Omega_{\sigma_3} \\ -\Lambda_{\sigma_3} \end{bmatrix}$$

where

$$\Omega_{\sigma_3} = \tfrac{\alpha(\alpha-1)\nu\bar{\delta}(1-\chi)}{(1+\chi i_d)^2}\left(\tfrac{1-\chi}{\chi}\left[(1-\sigma_1)d_3 + \tfrac{(1-\sigma_3)\chi\bar{\delta}}{1+\chi i_d}\right]\right)^{\alpha-2}, \quad \Lambda_{\sigma_3} = 0.$$

Using Cramer's rule gives following comparative statics results.

$$\frac{\partial i_d}{\partial \sigma_3} = \frac{\begin{vmatrix} -\Omega_{\sigma_3} & \Omega_{d_3} \\ -\Lambda_{\sigma_3} & \Lambda_{d_3} \end{vmatrix}}{\begin{vmatrix} \Omega_{i_d} & \Omega_{d_3} \\ \Lambda_{i_d} & \Lambda_{d_3} \end{vmatrix}} > 0, \quad \frac{\partial d_3}{\partial \sigma_3} = \frac{\begin{vmatrix} \Omega_{i_d} & -\Omega_{\sigma_3} \\ \Lambda_{i_d} & -\Lambda_{\sigma_3} \end{vmatrix}}{\begin{vmatrix} \Omega_{i_d} & \Omega_{d_3} \\ \Lambda_{i_d} & \Lambda_{d_3} \end{vmatrix}} > 0$$

Since $d_2 = d_3 + \frac{\chi\bar{\delta}}{1+\chi i_d}$, $\partial d_2/\partial\sigma_3 > 0$. While $\partial i_d/\partial\sigma_3 > 0$, we still have downward sloping demand for reserves as below:

$$\frac{1+i}{1+i_d} - 1 - \alpha\nu\left(\frac{1-\chi}{\chi}r\right)^{\alpha-1} - (k + i_d - i_r)\left(\frac{\chi}{1-\chi}\right) = 0$$

implying $\partial r/\partial\sigma_3 < 0$ where $r = \sigma_2 d_2 + \sigma_3 d_3$ with $\sigma_2 d_2 > \sigma_3 d_3$. Even though both d_2 and d_3 are increasing in σ_3, which have indirect positive impact on r, direct effect of σ_3 dominates those indirect impacts.

■

Appendix B Additional Results

B.1 Chow Test

Figure 1 includes the Chow test for structural breaks. The test result reported in the bottom-left panel of Figure 1 is implemented by estimating following regression.

$$\begin{aligned}\text{Money multiplier}_t =& \beta_0 + \beta_1(\text{RequiredReserves/Deposit})_t \\ &+ \mathbf{1}_{t\geq 1992Q2}[\gamma_0 + \gamma_1(\text{RequiredReserves/Deposit})_t] \\ &+ \mathbf{1}_{t\geq 2008Q4}[\delta_0 + \delta_1(\text{RequiredReserves/Deposit})_t] + \epsilon_t\end{aligned}$$

Table 7a reports F-statistics which are obtained by testing $\gamma_0 = \gamma_1 = \delta_0 = \delta_1 = 0$.

The Chow test in the bottom-right panel of Figure 1 is implemented by estimating following regression.

$$\begin{aligned}\text{Money multiplier}_t =& \beta_0 + \beta_1(\text{Currency/Deposit})_t \\ &+ \mathbf{1}_{t\geq 2008Q4}[\delta_0 + \delta_1(\text{Currency/Deposit})_t] + \epsilon_t\end{aligned}$$

Table 7b reports F-statistics is obtained by testing $\delta_0 = \delta_1 = 0$. The regression estimates and the Chow test results are summarized at the below table.

Table 7: Chow test for structural breaks

(a) Currency deposit ratio

Dependent Variable: Money Multiplier	
CD	-1.301^{***}
	(0.027)
CD $\times$ $\mathbf{1}_{t\geq 2008Q4}$	-52.018^{***}
	(4.995)
$\mathbf{1}_{t\geq 2008Q4}$	3.061^{***}
	(0.409)
Constant	3.159^{***}
	(0.015)
Obs.	228
R^2	0.974
DF for numerator	2
DF for denominator	224
F Statistic for Chow test	1245.69
F Statistic for 1% sig. level	4.70
F Statistic for 0.1% sig. level	7.13

(b) Require reserve ratio

Dependent Variable: Money Multiplier	
RR	-0.601
	(0.365)
RR $\times$ $\mathbf{1}_{t\geq 1992Q2}$	132.279^{***}
	(0.031)
RR $\times$ $\mathbf{1}_{t\geq 2008Q4}$	-147.943^{***}
	(8.574)
$\mathbf{1}_{t\geq 1992Q2}$	9.091^{***}
	(0.557)
$\mathbf{1}_{t\geq 2008Q4}$	0.074^{***}
	(0.611)
Constant	2.813^{***}
	(0.053)
Obs.	228
R^2	0.963
DF for numerator	4
DF for denominator	222
F Statistic for Chow test	1711.32
F Statistic for 1% sig. level	3.40
F Statistic for 0.1% sig. level	4.79

Notes: Newy-West standard errors are in parentheses. ***, **, and * denote significance at the 1, 5, and 10 percent levels, respectively. Degree of freedom is denoted by DF.

B.2 Unit Root and Cointegration Test

Column (1) and (2) in Table 5 includes the canonical cointegrating regression estimates since all three series - R/Y, UC/Y, and 3 month treasury rate - have unit roots and cointegrated both for the data and the model-implied series. This section reports unit root tests and cointegration tests and sensitivity check using federal funds rates.

Table 8: Johansen test for cointegration

(a) UC/Y, Tbill3 and R/Y (Data)

Max rank	$\lambda_{trace}(r)$	5% CV	1% CV
0	39.5289	29.68	35.65
1	**6.3521**	15.41	20.04
2	1.7359	3.76	6.65
Max rank	$\lambda_{max}(r, r+1)$	5% CV	1% CV
0	33.1768	20.97	25.52
1	**4.6162**	14.07	18.63
2	1.7359	3.76	6.65

(b) UC/Y, Tbill3 and R/Y (Model)

Max rank	$\lambda_{trace}(r)$	5% CV	1% CV
0	48.3176	29.68	35.65
1	**9.8317**	15.41	20.04
2	3.3576	3.76	6.65
Max rank	$\lambda_{max}(r, r+1)$	5% CV	1% CV
0	38.4859	20.97	25.52
1	**6.4741**	14.07	18.63
2	3.3576	3.76	6.65

(c) UC/Y, ffr and R/Y (Data)

Max rank	$\lambda_{trace}(r)$	5% CV	1% CV
0	42.2554	29.68	35.65
1	**6.1539**	15.41	20.04
2	1.7615	3.76	6.65
Max rank	$\lambda_{max}(r, r+1)$	5% CV	1% CV
0	36.1015	20.97	25.52
1	**4.3924**	14.07	18.63
2	1.7615	3.76	6.65

(d) UC/Y, ffr and R/Y (Model)

Max rank	$\lambda_{trace}(r)$	5% CV	1% CV
0	47.4812	29.68	35.65
1	**9.7295**	15.41	20.04
2	3.1840	3.76	6.65
Max rank	$\lambda_{max}(r, r+1)$	5% CV	1% CV
0	37.7517	20.97	25.52
1	**6.5455**	14.07	18.63
2	3.1840	3.76	6.65

Table 9: Unit root test and additional CCR estimates

(a) Unit root test

	Phillips-Perron test	
	$Z(\rho)$	$Z(t)$
`Tbill3`	−7.683	−1.967
`ffr`	−8.683	−2.121
UC/Y	−0.315	−0.450
R/Y(Data)	−1.735	−2.240
R/Y(Model)	−1.100	−1.877
Δ`Tbill3`	−24.363***	−4.514***
Δ`ffr`	−25.127***	−4.747***
$\Delta UC/Y$	−24.200***	−4.201***
$\Delta R/Y$(Data)	−25.854***	−4.247***
$\Delta R/Y$(Model)	−31.710***	−4.993***

(b) Canonical cointegrating regression

Dependent Variable:	Reserves/GDP (1968-2007)
Unsecured Credit/GDP	−0.122***
	(0.004)
Federal Funds Rate	−0.064***
	(0.009)
Constant	3.058***
	(0.095)
Obs.	40
R^2	0.854
adj R^2	0.846
Long run S.E.	0.141

Notes: All series are demeaned before implementing the unit root test because the magnitude of the initial value can be problematic, as pointed out by Elliott and Müller (2006) and Harvey, Leybourne and Taylor (2009). ***, **, and * denote significance at the 1, 5, and 10 percent levels, respectively.

B.3 Robustness I: Other Specifications and M2

This section summarizes alternative parameterization results. For robustness, I examine how results are sensitive with respect to different α and σ_3 while keeping other parameters are same. In Model 3 and 4, I set α as 1.8 and 2.2. For Model 1 and 2, I set α as 1.8 and 2.2 and $\sigma_3 = 0.3$. In addition to those I repeat the calibration using M2 instead M1.[15] For M2 exercise, I recalculated required reserves ratio by computing (required reserves)/(deposit component of M2).

Table 10: Alternative parametrizations

	Data	Baseline	Model 1	Model 2	Model 3	Model 4	M2
External Parameters							
α		2	1.8	2.2	1.8	2.2	2
σ_3		0.4783	0.3	0.3	0.4783	0.4783	0.4783
Calibration targets							
avg. retail markup	1.384	1.384	1.386	1.383	1.384	1.383	1.387
avg. C/Y	0.044	0.044	0.044	0.044	0.044	0.044	0.044
avg. R/Y	0.016	0.016	0.016	0.016	0.016	0.016	0.011
semi-elasticity of C/Y	-3.716	-3.724	-3.720	-3.729	-3.724	-3.729	-3.019
avg. C/D	0.529	0.564	0.574	0.557	0.564	0.557	
avg. UC/DM	0.387	0.378	0.379	0.377	0.378	0.377	
avg. C/D (M2)	0.090						0.103
avg. UC/DM (M2)	0.159						0.175

Note: C, R, DM, UC, Y denote currency in circulation, reserves, DM transactions, unsecured credit and nominal GDP, respectively.

Table 10 compares how model moments are changing depending on different parameterization. The table shows that the results are not very sensitive since changing α and σ_3 provides similar moments that can be matched with target moments. Figure 14 and 13 compares the model fits of model-generated series under different parameterization. The model-generated series also show similar patterns for M1 Money multiplier, excess reserve ratio and currency deposit ratio. The exercise using M2 also can fit the target moments as shown in Table 10. As shown in Figure 14, model-generated series using M2 based calibration can mimic the overall evolution of each series: M2 Money multiplier, excess reserve ratio, and currency deposit ratio. Using M2 gives improved model fit in levels compared to calibrated example using M1.

[15] For M2 model, calibrated parameters are $\theta = 0.51$, $\sigma_1 = 0.089$, $A = 1.04$, $B = 1$, $k = 0.003$, $\gamma = 0.434$ $\nu = 0.0017$ and the federal funds rate is used as nominal interest rate. semi-elasticity of C/Y for federal funds rate is -3.020

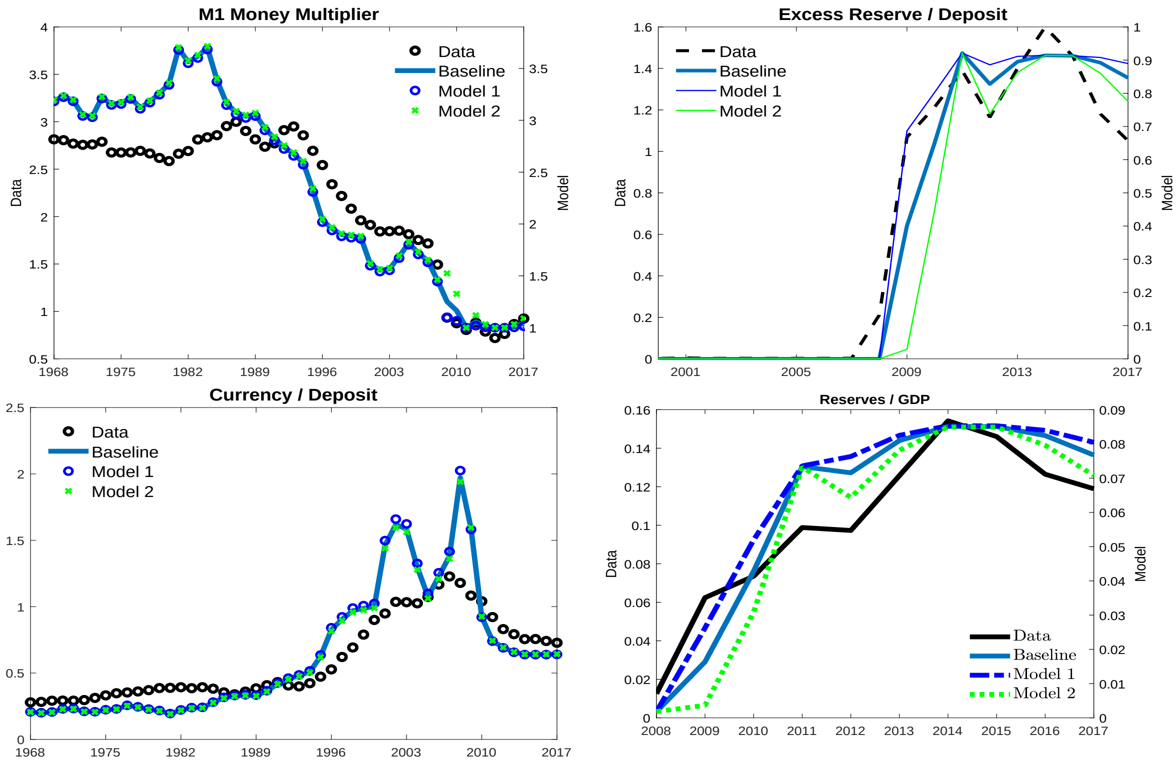

Figure 13: Model fit with different specifications

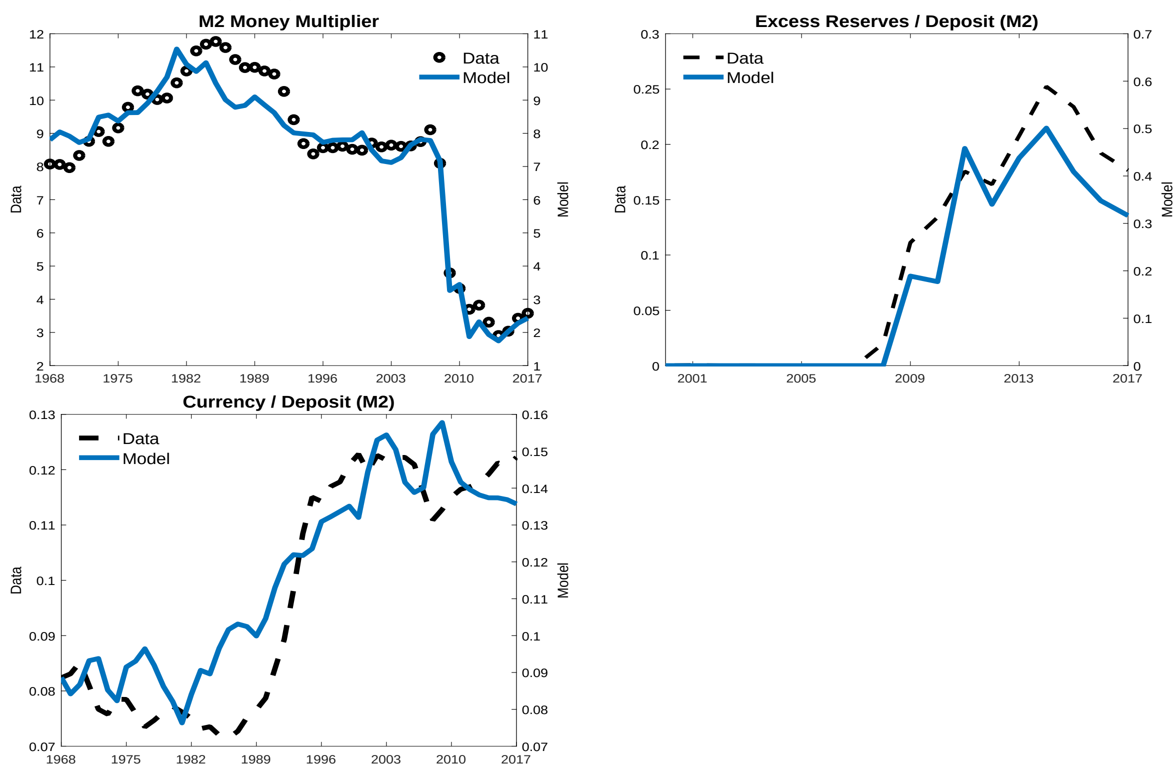

Figure 14: Model fit with M2

B.4 Robustness II: Different Measure of Monetary Policy

Different papers have use different series as monetary instruments for fitting the monetary model to the money demand. For example, Lucas (2000) and Lagos and Wright (2005) use commercial paper rate. Bethune et al. (2020) uses 3 month treasury bill rate. New Keynesian literature usually use federal funds rate (e.g, Christiano, Eichenbaum and Evans, 2005 Smets and Wouters, 2007, Christiano, Eichenbaum and Trabandt, 2016) or 3 month treasury bill rate (e.g., Ireland, 2011) as measure of monetary policy while those models don't fit to the money demand. This section checks the robustness of the main quantitative results by refitting the model to the data using different measure of monetary policy target: commercial paper rate, 3 month T-bill rate, inflation rate. For the inflation rate, I use core PCE inflation rate which have been targeted by the Federal Reserves.

Table 11: Parametrizations with different measure of monetary policy

Interest/Inflation rate	3 Month T-bill		Federal Funds		CP		Core PCE	
	Data	Model	Data	Model	Data	Model	Data	Model
Targets								
avg. retail markup	1.384	1.384	1.384	1.384	1.384	1.384	1.384	1.384
avg. C/Y	0.044	0.044	0.044	0.044	0.044	0.044	0.044	0.044
avg. R/Y	0.016	0.016	0.016	0.016	0.016	0.016	0.016	0.016
avg. C/D	0.529	0.564	0.529	0.531	0.529	0.554	0.529	0.551
avg. UC/DM	0.387	0.378	0.387	0.373	0.387	0.376	0.387	0.375
semi-elasticity of C/Y	-3.716	-3.724	-3.020	-3.012	-3.454	-3.440	-4.258	-4.220
Parameter								
bargaining power, θ		0.454		0.512		0.476		0.423
enforcement cost level, ν		0.020		0.019		0.016		0.016
DM1 matching prob, σ_1		0.189		0.184		0.189		0.201
deposit operating cost, k		0.002		0.002		0.002		0.002
DM utility level, A		0.618		0.598		0.611		0.642
DM utility curvature, γ		0.398		0.427		0.408		0.378

Note: C, R, DM, UC, Y denote currency in circulation, reserves, DM transactions, unsecured credit and nominal GDP, respectively.

Table 11 compares how model moments are changing depending on different parameterization using different measures of monetary policy. The table shows that the results are not very sensitive since changing measure of monetary policy provides similar moments that can be matched with target moments. Figure 15 and 15 compares the model fits of model-generated series under different parameterization. The model-generated series also show similar patterns for M1 Money multiplier, excess reserve ratio and currency deposit ratio except for the exercise using inflation rate.

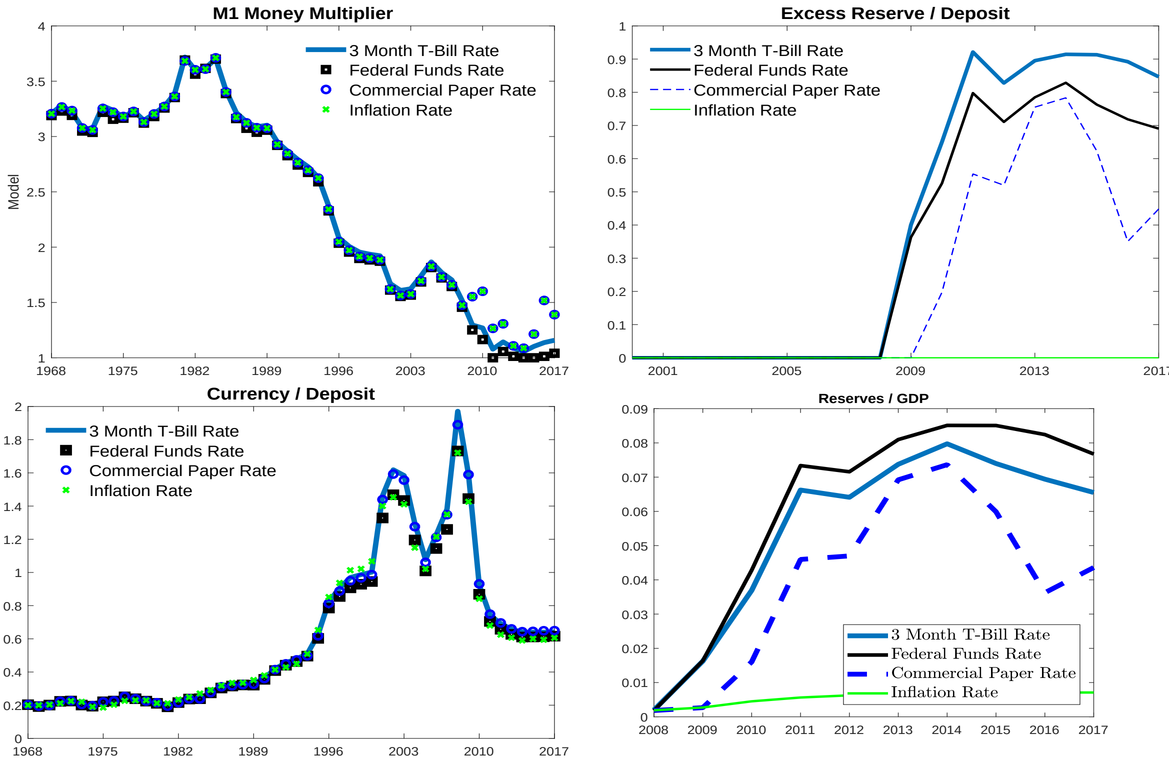

Figure 15: Model fit with measure of monetary policy

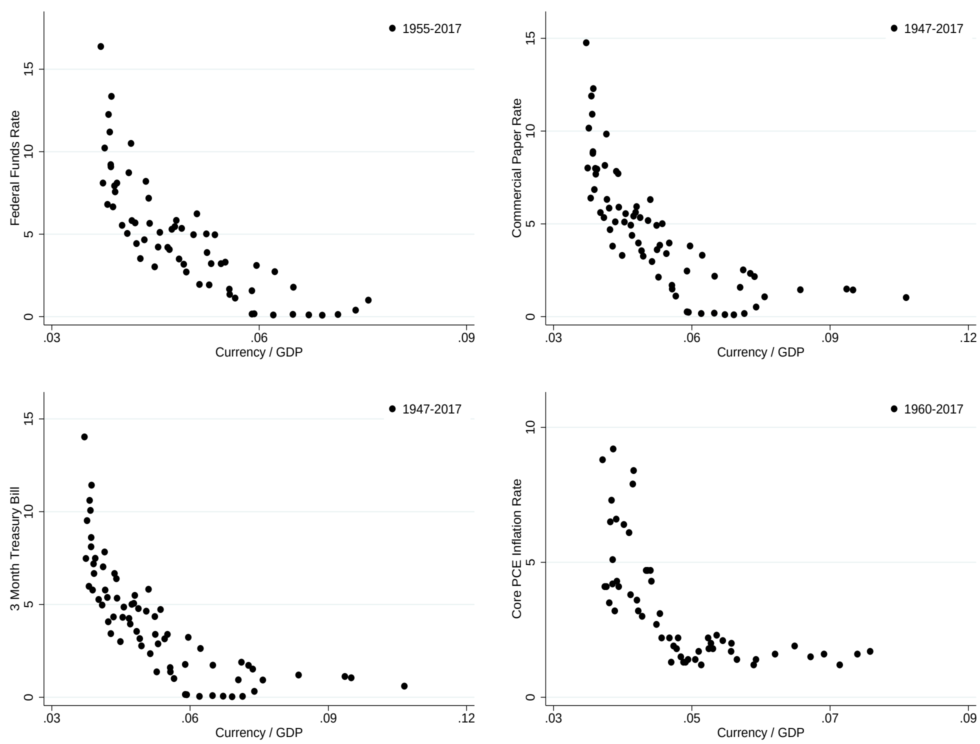

Figure 16: Money demand for currency