Price Competition Online: Platforms vs. Branded Websites

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Abstract

The focus of this theoretical study is price competition when some firms operate their own branded websites while others sell their products through an online platform, such as Amazon Marketplace. On one hand, selling through Amazon expands a firm's reach to more customers, but on the other, starting a website can help the firm to increase the perceived value of its product, that is, to build brand equity. In the short run the composition of firms is fixed, whereas in the long run each firm chooses between Amazon and its own website. I derive the equilibrium prices and profits, analyze the firms’ behavior in the long run, and compare the equilibrium outcome with the social optimum. Comparative statics analysis reveals some interesting results. For example, I find that the number of firms that choose Amazon may go down in response to an increase in the total number of firms. A pure-strategy Nash equilibrium may not exist; I show that price dispersion among the firms of the same type is more likely in less concentrated markets and/or when the increase in the perceived value of the product is relatively small.

Key words: pricing, competition, platforms, online marketplace, Amazon, brand equity

JEL codes: C72, D43, L11, L13, M31

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1 Introduction

Online marketplaces, where consumers can purchase from a variety of retailers on one website, have become an integral part of many eCommerce sellers’ businesses. They represent an opportunity for retailers to expand their reach to more customers, both nationally and internationally. The most prominent examples of online marketplaces are eBay, Amazon Marketplace, Google, Flipkart (India), and Alibaba and Taobao (China). Selling through an online marketplace generally comes at a cost to retailers. For example, Amazon charges its third-party retailers a referral fee for each sale. The retailers who use Amazon’s fulfillment service pay extra. Yet, over two million sellers are willing to pay these fees to gain access to Amazon’s more than 300 million customers.\(^1\) According to Internet Retailer, in the first half of 2019 over half of the products sold on Amazon were from the marketplace sellers.\(^2\)

Selling through an online marketplace (henceforth, I will collectively refer to such marketplaces as Amazon) seems like a good idea. All you have to do is source your product and Amazon handles the rest by providing you with an endless stream of consumer traffic. The catch is that people who shop on Amazon think they are buying from Amazon even though it is your product they are buying. This means that with Amazon, you have zero brand equity. What is worse, you do not even have access to your customer list. On the other hand, if you start your own website, you can compile a database of your customers, and implement a variety of marketing strategies. You can create a special promotion for all customers who purchased a specific product. You can award repeated customers. You can build trust by sending your customers useful content. You can take advantage of customer acquisition platforms, such as Google Shopping and Google AdWords, Bing Shopping and Bing Ads, Facebook, Instagram, Pinterest and YouTube, while simultaneously building your brand equity.

Lululemon, Timberland, Under Armour, and REI are examples of companies whose main online distribution channel is their website. In the same sports apparel industry there are firms that gravitated towards Amazon: OYANUS, Running Girl, and Camelsports. In bedding and home improvement industry, Crane & Canopy operates its own website, while Utopia Bedding is among the top 100 sellers on Amazon Marketplace.\(^3\)

In this theoretical study, I consider a setting in which firms can sell their products through an online marketplace, such as Amazon, or through their own websites. On one hand, selling through Amazon expands a firm’s reach to more customers, but on the other, starting a website can help the firm to increase the perceived value of its product, that is, to build brand equity. My research questions are: (i) Should a firm sell its product through Amazon or through its

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\(^1\)https://www.statista.com/statistics/476196/number-of-active-amazon-customer-accounts-quarter/
\(^3\)https://www.marketplacepulse.com/amazon/top-amazon-usa-sellers
own branded website, and at what price? (ii) What is the effect of market concentration on the number of firms that choose Amazon in equilibrium? (iii) Is there too many or too few firms on Amazon in equilibrium compared to the social optimum?

To address these questions, I use the spokes model of spatial competition proposed by Chen and Riordan (2007). The spokes model extends the linear city model (Hotelling, 1929) to arbitrary numbers of product varieties and of firms. Unlike the circular city model (Salop, 1979), competition is non-localized. The preference space consists of $N$ lines, indexed by $i$, that start at the same central point (resembling the spokes on a bicycle wheel). Variety $i$ is located at the extreme end of spoke $i$. There are $n \leq N$ firms, each producing a single variety. Consumers are uniformly distributed on the network of spokes and incur transportation costs – utility losses from imperfect matching – as they travel to a firm to purchase the firm’s product. For a consumer located on spoke $i$ variety $i$ is her first preferred brand, and each of the other $N-1$ varieties is equally likely to be her second preferred brand. The consumer has value $v$ for one unit of either her first or second preferred brand, and zero value for other brands.

In my variation of the spokes model there are two types of firms, a-firms that sell through Amazon and w-firms that sell through their own websites. Their numbers are $n_a$ and $n - n_a$, respectively. Consider a consumer located on spoke $i$ and suppose that an a-firm (a w-firm) is located at its extreme end. The consumer derives value $v$ (value $v + \Delta$) from consuming one unit of variety $i$. As in the spokes model, each of the other $N - 1$ varieties is equally likely to be her second preferred brand. However, the consumer is only aware of the varieties that are sold on Amazon. Put differently, if you are an a-firm, every consumer can “see” your product. If you are a w-firm selling variety $i$ through your own branded website, only consumers located on spoke $i$ are aware of your existence. The good thing is that they are fans of your product, which is captured by the brand equity parameter $\Delta > 0$.

The firms set their prices simultaneously. In the short run (Section 3) the composition of firms, $n_a$, is fixed. I first show that in any symmetric pure-strategy Nash equilibrium the difference between the price charged by w-firms and the price charged by a-firms must exceed $\Delta$. I then calculate the equilibrium prices. I find that the prices do not depend on the composition of firms when the number of a-firms is small, and that both prices are decreasing in $n_a$ when $n_a$ is intermediate. The equilibrium involves mixed strategies when the number of a-firms is large, which means price dispersion among firms of the same type.

In the long run (Section 4) each firm can choose whether to build its own website or join Amazon Marketplace. The number of a-firms becomes endogeneous. Comparative statics analysis reveals some interesting results. For example, the equilibrium number of a-firms may go down as the total number of firms increases. I also show that price dispersion among firms of the same type is more likely for smaller values of $n$ and/or $\Delta$. 

From the social planner’s perspective, a branded website increases the perceived value of the product; the drawback is that consumers who lack their first choices and whose second choice is a variety produced by a w-firm end up not purchasing. I show that the social optimum will be achieved if $\Delta$ is so high that in equilibrium all firms choose to operate their own websites. In all other situations the equilibrium number of firms on Amazon tends to be excessive for higher values of $n$, and deficient in less concentrated markets.

1.1 Related Literature

The primary focus of my paper is price competition between firms, when some sell through an online marketplace while the others run their own websites. The existing theoretical papers on platform-based Internet retailing address complementary research questions. Jiang, Jerath, and Srinivasan (2011) investigate how a platform owner such as Amazon learns from the third-party retailers’ early sales which of the products it should procure and sell directly and which it should leave for others to sell. The authors show that such “cherry-picking” of the successful products gives the high-demand retailers incentives to lower their sales by reducing the level of service.

Abhishek, Jerath, and Zhang (2016) analyze online retailers’ incentives to adopt the agency selling format, under which they allow manufacturers direct access to their customers for a fee. Amazon Marketplace, Marketplace at Sears, Taobao and Flipkart serve as examples of the agency selling format. The authors find that agency selling leads to lower prices and is more efficient than reselling. However, whether e-tailers end up giving control over prices to manufacturers depends on the degree of competition between e-tailers, as well as on the extent of the demand spillover between the electronic channel and the traditional brick-and-mortar channel. The strategic choice by an intermediary between agency selling and reselling is also studied in Hagiu and Wright (2015a). Drivers of the optimal intermediation mode that the authors analyze include the relative importance of the suppliers’ versus the intermediary’s private information, whether the products are long tail or short tail, and the presence of spillovers across products generated by marketing activities.

Ryan, Sun, and Zhao (2012) consider a setting in which a marketplace firm (Amazon) operates an online marketplace (Amazon Marketplace). There is a single online firm that can choose to pay a fee and sell its product on Amazon Marketplace. Selling through Amazon expands the firm’s customer base. A key question for the firm is whether to sell through Amazon Marketplace and at what price. (The same question the firms in my model face.) The authors are also interested in the marketplace firm’s decision whether to sell a competing product and what fee to set.

Mantin, Krishnan, and Dhar (2014) is another study of dual-format retailing. The authors develop a bargaining model between a retailer and a manufacturer. The retailer sells the man-
ufacturer’s product. There is a competitive fringe of third-party retailers that sell a substitute product of a lower quality. The third-party retailers pay the retailer a per-unit marketplace transaction fee. The authors show that by committing to an active online marketplace, the retailer improves its bargaining position in negotiations with the manufacturer.

My paper is also related to the recent studies that build on the spokes model (Caminal and Claiici, 2007, Amaldoss and He, 2010, Caminal and Granero, 2012, Granero, 2013, Reggiani, 2014) and to the theoretical studies of multi-sided platforms, since an online marketplace is a type of a two-sided platform (Cailalud and Jullien, 2003, Rochet and Tirole, 2003, Parker and Van Alstyne, 2005, Armstrong, 2006, Hagiu and Wright, 2015b, among others).

2 Model Setup

Consider a market along the lines of Chen and Riordan (2007). There are $N$ possible varieties of a differentiated product, indexed by $i = 1, \ldots, N$. The preference space consists of $N$ lines (spokes) of length $1/2$, also indexed by $i$, that start at the same central point. Variety $i$ is located at the extreme end of spoke $i$. Consumers of total mass $N/2$ are uniformly distributed over the spokes network. For a consumer located on spoke $i$ variety $i$ is her first preferred brand, and each of the other $N - 1$ varieties is equally likely to be her second preferred brand. The consumer has value $v$ for one unit of her first or second preferred brand, and zero value for other brands.

In Chen and Riordan’s setup there are $n \leq N$ identical firms, each producing a single variety. A consumer located on spoke $i$ at distance $x_i$ from variety $i$ obtains utility $v - tx_i$ from consuming it, where $t$ is the unit transportation cost. The distance to any other variety is $1/2 - x_i + 1/2 = 1 - x_i$. This is because the consumer covers $1/2 - x_i$ units of distance to get to the center of the spokes network and then $1/2$ units of distance to get to variety $j$ $\neq i$. Thus, the consumer obtains utility $v - t(1 - x_i)$ from consuming her second preferred brand.

I consider a variation of the spokes model. In my setup there are two types of firms, a-firms that sell through an online marketplace platform (henceforth, Amazon) and w-firms that run their own websites. While Amazon provides firms with endless stream of consumer traffic, starting a website helps a firm to increase the perceived value of its product, that is, to build brand equity. Let $n_a$ denote the number of a-firms, then the number of w-firms will be $n - n_a$. If you are an a-firm, everyone can “see” you. If you are a w-firm selling variety $i$ through your own branded website, only consumers located on spoke $i$ are aware of your existence. The good thing is that they are fans of your product, which is captured by the brand equity parameter

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4Chen and Riordan assume that each consumer only cares about two brands to assure the existence of a symmetric pure-strategy equilibrium in prices.
Figure 1: Spokes Model with $N = 12$, $n = 6$ and $n_a = 4$

$\Delta > 0$ that transforms the aforementioned value $v$ into $v + \Delta$.

Figure 1 depicts a radial network with twelve spokes (lines $l_1, l_2, \ldots l_{12}$), four a-firms and two w-firms. The firms that sell through Amazon are firms $A_1, A_5, A_6$ and $A_{10}$: $A_1$ produces variety 1, $A_5$ variety 5, $A_6$ variety 6 and $A_{10}$ variety 10. The firms that run their own websites are firms $W_3$ and $W_8$: $W_3$ produces variety 3 and $W_8$ produces variety 8.

Consider a consumer located on $l_1$ at distance $x_1$ from variety 1. Variety 1 is her first preferred brand; she obtains utility $v - t x_1$ from consuming it. The consumer’s second preferred brand is variety $j \neq 1$ chosen by nature with probability $1/(N - 1)$. Too bad if $j = 2, 4, 7, 9, 11$ or 12, as these varieties are not produced by any of the firms. If $j = 3$ or 8, that is not good either, because our consumer only sees varieties offered on Amazon. Only if $j = 5, 6$ or 10 the consumer’s second preferred brand is available to her; she obtains utility $v - t (1 - x_1)$ from consuming it.

Next, consider a consumer located on $l_2$ at distance $x_2$ from variety 2. Unfortunately for this consumer, variety 2 is not produced by any of the firms. The consumer’s second preferred brand is available to her if it is sold on Amazon by firm $A_j$, $j = 1, 5, 6$ or 10; she obtains utility $v - t (1 - x_2)$ from consuming it. Finally, consider a consumer located on $l_3$ at distance $x_3$ from variety 3. Variety 3 is her first preferred brand; she obtains utility $v + \Delta - t x_3$ from consuming it. The consumer’s second preferred brand is available to her if it is sold by firm $A_j$, $j = 1, 5, 6$ or 10; she obtains utility $v - t (1 - x_3)$ from consuming it.

I assume that each firm’s variable production cost equals $c$, regardless of whether it sells
through its own website or through Amazon. The firms set their prices simultaneously. Consumers observe the prices and make their purchasing decisions.

3 Equilibrium Analysis

We will start the equilibrium analysis with consumers, then analyze the firms’s pricing decisions.

3.1 Consumer Demand

Consider a consumer whose first preferred brand is variety $i$ and whose second preferred brand is variety $j$. Suppose variety $i$ is sold by a $w$-firm (firm $W_i$) and variety $j$ is sold by an $a$-firm (firm $A_j$). The consumer compares her payoff from purchasing from firm $W_i$,

$$v + \Delta - tx_i - p_i,$$

with that from purchasing from firm $A_j$,

$$v - t(1 - x_i) - p_j,$$

where $p_i$ and $p_j$ are the prices charged by the firms. If $p_i \leq p_j + \Delta$, the consumer will purchase variety $i$ irrespective of her location on $l_i$. If $p_i > p_j + \Delta$, she will purchase from firm $W_i$ if and only if

$$x_i \leq \frac{1}{2t} (t + \Delta - p_i - p_j) .$$

Next, suppose that both varieties $i$ and $j$ are sold on Amazon (by firms $A_i$ and $A_j$). If $p_i \leq p_j$, the consumer will purchase from firm $A_i$ irrespective of her location on $l_i$. If $p_i > p_j$, the consumer compares

$$v - tx_i - p_i$$

with

$$v - t(1 - x_i) - p_j,$$

and will purchase from firm $A_i$ if and only if

$$x_i \leq \frac{1}{2t} (t - p_i + p_j) .$$

The consumer’s purchasing decision is trivial when variety $i$ or variety $j$ is available to her but not both. That is, when either variety $j$ is not available on Amazon or variety $i$ is not produced by any of the firms.
We now derive the demand that the firm located at the end of spoke $i$ faces, be it a firm that operates its own website (firm $W_i$), or a firm that sells its product through Amazon (firm $A_i$). Consider consumers on $l_i$ and $l_j$ for whom both variety $i$ and variety $j$ are the two desired brands. There are two types of marginal consumers. If variety $i$ is produced by a w-firm and variety $j$ is produced by an a-firm, the marginal consumer consumer is located on $l_i$ at distance

$$\hat{x}(p_i, p_j) = \max \left\{ \min \left\{ \frac{1}{2t} (t + \Delta - p_i + p_j), \frac{1}{2} \right\}, 0 \right\}$$

from variety $i$. If variety $i$ and variety $j$ are both produced by a-firms, the marginal consumer is located at distance

$$\hat{y}(p_i, p_j) = \max \left\{ \min \left\{ \frac{1}{2t} (t - p_i + p_j), 1 \right\}, 0 \right\}$$

from variety $i$. It is easy to see that the marginal consumer is located on $l_i$ (\(\hat{y} < 1/2\)) if $p_i < p_j$, and on $l_j$ (\(\hat{y} > 1/2\)) if $p_i > p_j$.

Firm $W_i$ sells to two groups of consumers: consumers on $l_i$ who have an alternative available on Amazon, and those who do not. Summing up over these groups yields firm $W_i$’s total demand:

$$q_i = \frac{1}{N-1} \sum_{j: \text{all a-firms}} \hat{x}(p_i, p_j) + \frac{1}{2} \frac{N - n_a - 1}{N - 1}. \quad (3)$$

Firm $A_i$ sells to four groups: (i) consumers whose desired brands are both sold on Amazon, one of which is variety $i$, (ii) consumers whose first choice is produced by a w-firm and whose second choice is variety $i$, (iii) consumers on $l_i$ who do not have an alternative available on Amazon, and (iv) consumers who lack their first choices but for whom variety $i$ is their second preferred brand. Summing up yields firm $A_i$’s total demand:

$$q_i = \frac{1}{N-1} \sum_{j: \text{all a-firms but } A_i} \hat{y}(p_i, p_j) + \frac{1}{2} \frac{N - n_a - 1}{N - 1} + \frac{1}{2} \frac{N - n_a}{N - 1} + \frac{1}{2} \frac{N - n}{N - 1}. \quad (4)$$

When deriving the demand functions (3) and (4), I assumed that the prices are not prohibitively high so that all consumers whose desired varieties are available purchase. I did so for the ease of exposition. The situations where some consumers do not purchase because the prices are too high will not arise in equilibrium.
3.2 Pricing

I will focus on symmetric pure-strategy Nash equilibria in which a-firms charge the same price $p^*_a$ and w-firms charge the same price $p^*_w$. To keep the model tractable and the results focused, I will assume

$$c + 2 \frac{N-1}{n-1} t \leq v \leq c + \left(2 \frac{N-1}{n-1} + \frac{1}{2} \frac{12N-n-1}{N-n}\right) t$$

throughout the paper. Notice that the upper limit placed on $v$ is $+\infty$ when $n = N$. (This assumption will be discussed in more detail after Lemma 2.)

Let us first solve the game for the two extreme values of $n_a$. When all firms operate their own websites ($n_a = 0$), they are essentially monopolists. Because $v$ is relatively high under the assumption (5), each w-firm finds it optimal to serve all of its fans – consumers located on the same spoke – by charging

$$p^*_w = v + \Delta - \frac{t}{2}.$$

We have our first result:

**Lemma 1 (Equilibrium prices and profits when $n_a = 0$).** When all firms operate their own websites, each sets its price equal to

$$p^*_w = v + \Delta - \frac{t}{2},$$

serves consumers located on the same spoke, and earns the profit

$$\Pi^*_w = \frac{1}{2} \left( v + \Delta - \frac{t}{2} - c \right).$$

When all firms sell their products through Amazon ($n_a = n$), the equilibrium price is forced down by competition for the marginal consumers described in (2). The proof of Lemma 2 is relegated to the Appendix.

**Lemma 2 (Equilibrium prices and profits when $n_a = n$).** When all firms sell their products through Amazon, they set their prices equal to

$$p^*_a = c + \frac{2N-n-1}{n-1} t.$$ 

In equilibrium, each firm $A_i$ serves consumers located on spoke $i$ as well as those who lack their first choices but for whom variety $i$ is their second choice. The resulting profit is

$$\Pi^*_a = \frac{(2N-n-1)^2}{2(N-1)(n-1)} t.$$
Remark 1. The present model reduces to the spokes model when all firms sell their products through Amazon.

In the spokes model, the nature of the equilibrium differs for different regions in the parameter space (Chen and Riordan, 2007, pp. 903-905). The assumed parameter space (5) matches the region that corresponds to the normal oligopoly competition. Chen and Riordan call it normal because an increase in market concentration has the familiar effect of lowering the equilibrium prices and profits. As we see, both $p^*_a$ and $\Pi^*_a$ in Lemma 2 are decreasing in $n$.

What happens when $n_a$ takes intermediate values? The next lemma establishes that in any symmetric pure-strategy Nash equilibrium the difference between the price that w-firms charge and the price that a-firms charge must exceed $\Delta$.

Lemma 3 (The price difference $p^*_w - p^*_a$). In a symmetric pure-strategy Nash equilibrium $p^*_w - p^*_a > \Delta$.

That $p^*_w - p^*_a < \Delta$ cannot happen in equilibrium is easy to see. If a w-firm increases its price to $p^*_w + \Delta$, it will not lose any of its customers. Thus, the deviation is profitable. Now suppose $p^*_w - p^*_a = \Delta$. Under such prices each firm serves consumers located on the same spoke; a-firms also serve consumers who lack their first choices but whose second choices are available on Amazon. If an a-firm decreases its price, it will steal consumers from all the other firms (a-firms and w-firms). If the firm increases its price, it will lose some of its customers only to the other a-firms. Thus, there is a kink in demand. In the Appendix, I first derive the condition under which any deviation to a lower price is unprofitable:

$$p^*_a \leq c + \frac{2N - n - 1}{n - 1} t.$$ 

I then derive the condition that takes care of the firm’s incentives not to increase its price:

$$p^*_a \geq c + \frac{2N - n - 1}{n_a - 1} t.$$ 

Since

$$c + \frac{2N - n - 1}{n - 1} t < c + \frac{2N - n - 1}{n_a - 1} t$$

for any $n_a < n$, a price that satisfies both conditions does not exist. Therefore, $p^*_w - p^*_a = \Delta$ cannot happen in equilibrium.

In Proposition 1, I show that a unique symmetric pure-strategy Nash equilibrium exists when $n_a$ is below some threshold $n_a^{III}$. The nature of competition changes as $n_a$ crosses two other
thresholds, $n_{a}^{I}$ and $n_{a}^{II}$, with $n_{a}^{I} < n_{a}^{II} < n_{a}^{III}$. While the expression for $n_{a}^{I}$ is compact,

$$n_{a}^{I} = \frac{N - 1}{\frac{\Delta}{t} - \frac{1}{2}},$$

and $n_{a}^{II}$ is the solution to a quadratic equation,

$$n_{a}^{II} = \sqrt{\frac{\left(N + n - 3\right)\Delta + \left(N - 3\right)\Delta t}{{2t}}} - \frac{\sqrt{\left(N + n - 3\right)\Delta + \left(n - 1\right)\Delta t}}{{2t}} - (N - 1)(n - 1),$$

$n_{a}^{III}$ can only be defined implicitly. (Please see the proof of Proposition 1 in the Appendix.)

**Proposition 1** (Equilibrium prices when $n_{a} \in (0, n_{a}^{III})$). For any $n_{a} \in (0, n_{a}^{III})$, the pricing game has a unique symmetric pure-strategy Nash equilibrium in which a-firms charge the same price $p_{a}^{*}$ and w-firms charge the same price $p_{w}^{*}$.

(i) When $n_{a} \in (0, n_{a}^{I})$ (Region I), the prices are

$$p_{w}^{*} = v + \frac{\Delta}{2} \quad \text{and} \quad p_{a}^{*} = v - \frac{3\Delta}{2}.$$

(ii) When $n_{a} \in (n_{a}^{I}, n_{a}^{II})$ (Region II), the prices are

$$p_{w}^{*} = c + \frac{N - n_{a}^{I} - 1}{n_{a}^{I}} \quad \text{and} \quad p_{a}^{*} = c + \frac{N - n_{a}^{II} - 1}{n_{a}^{II}} - \Delta.$$

(iii) When $n_{a} \in (n_{a}^{II}, n_{a}^{III})$ (Region III), the prices are

$$p_{w}^{*} = c + \frac{N - n_{a}^{II} + \frac{(N - 1)(2n_{a} - 1)}{n_{a}^{II}}}{n_{a}^{III} - n_{a}^{II}} \quad \text{and} \quad p_{a}^{*} = c + \frac{3N - 2n_{a}^{II} + \frac{(N - 1)n_{a}^{II}}{n_{a}^{III}}}{3n_{a}^{III} - n_{a}^{II}} - \Delta.$$

**Remark 2.** When $n_{a} \in (n_{a}^{III}, n)$ (Region IV), the equilibrium is in mixed strategies.

Figure 2 plots the equilibrium prices against the number of a-firms; in this figure $c = 2$, $v = 7$, $t = 1$, $\Delta = 1$, $N = 100$, and $n = 90$. When the number of a-firms is small, w-firms adopt the same pricing strategy as in Lemma 1. A-firms focus on capturing consumers whose first choice is produced by a w-firm, so in Region I they set their prices equal to

$$p_{a}^{*} = p_{w}^{*} - t - \Delta = v - \frac{3\Delta}{2}.$$
In equilibrium each w-firm earns the profit

\[ \Pi_w^* = \frac{1}{2} \frac{N - n_a - 1}{N - 1} (p_w^* - c). \]  

(6)

Here 1/2 is the number of consumers located on the same spoke. Only those whose second preferred brand is not available on Amazon will purchase from the firm, which explains why 1/2 is multiplied by \((N - n_a - 1)/(N - 1)\). As to a-firms, each earns

\[ \Pi_a^* = \left( \frac{1}{2} + \frac{1}{2} \frac{N - n_a}{N - 1} \right) (p_a^* - c). \]  

(7)

In addition to consumers located on spoke \(i\), firm \(A_i\) sells to consumers whose first choice is produced by a w-firm and whose second choice is variety \(i\), hence the terms 1/2 and 1/2 \(\times (N - n_a)/(N - 1)\) in (7).

As \(n_a\) increases, the fraction of consumers that a w-firm loses to the firms on Amazon increases (the term \((N - n_a - 1)/(N - 1)\) in (6) decreases). When \(n_a\) crosses the threshold
When \( n_a \in (n_a^{II}, n_a^{III}] \) (Region III), the equilibrium prices are forced down by competition for the marginal consumers in (1) as well as in (2). While the price difference \( p^*_w - p^*_a \) is constant (equals \( t + \Delta \)) in Regions I and II, in Region III it decreases from \( t + \Delta \) at \( n_a = n_a^{II} \) to some number between \( t + \Delta \) and \( \Delta \) at \( n_a = n_a^{III} \).

Finally, when \( n_a \in (n_a^{III}, n) \) (Region IV), a symmetric pure-strategy Nash equilibrium does not exist. The equilibrium is in mixed strategies, which means firms of the same type charge different prices. This result is important because it rationalizes price dispersion observed in online markets. Recall that \( n_a = n \) yields a symmetric equilibrium in pure strategies (Lemma 2). As we see, a small number of w-firms shudder this symmetry.

Equilibrium profits in Regions I-III are calculated easily from Proposition 1. Figure 3 plots the equilibrium profits against the number of a-firms for the same parameter values Figure 2 does: \( c = 2, v = 7, t = 1, \Delta = 1, N = 100, \) and \( n = 90 \). Both \( \Pi^*_w(n_a) \) and \( \Pi^*_a(n_a) \) are decreasing functions of \( n_a \), as expected.

**Corollary 1** (Equilibrium profits when \( n_a \in (0, n_a^{III}] \)). The profits of each w-firm and each a-firm in the unique symmetric pure-strategy Nash equilibrium are:

\[
\Pi^*_w(n_a) = \frac{N - n_a - 1}{2(N - 1)} \left( v + \Delta - t - c \right) \quad \text{and} \quad \Pi^*_a(n_a) = \frac{2N - n_a - 1}{2(N - 1)} \left( v - 3t - c \right)
\]

in Region I.

\[
\Pi^*_w(n_a) = \frac{(N - n_a - 1)^2}{2(N - 1)n_a} t \quad \text{and} \quad \Pi^*_a(n_a) = \frac{2N - n_a - 1}{2(N - 1)} \left( \frac{N - 2n_a - 1}{n_a} t - \Delta \right).
\]

in Region II.

\[
\Pi^*_w(n_a) = \frac{n_a}{2(N - 1)t} \left( \frac{N - n + \frac{(N-1)(2n-1)}{n_a}}{3n - n_a - 1} \right)^2
\]

and

\[
\Pi^*_a(n_a) = \frac{n - 1}{2(N - 1)t} \left( \frac{3N - 2n - 1 + \frac{(N-1)n}{n_a}}{3n - n_a - 1} \right)^2
\]

in Region III.

How does an increase in \( n_a \) affect consumers? First, an increase in \( n_a \) enables consumers whose first choice is not produced and whose second choice was produced by a w-firm and
now is produced by an a-firm to obtain a positive surplus. Second, when a w-firm becomes an a-firm, the consumers located on the same spoke no longer receive $\Delta$ when they purchase from the firm. Because the new price is lower by more than $\Delta$ (Lemma 3), they still benefit. There is also a group of consumers who switch from w-firms to this converted firm. They obviously benefit (would not switch otherwise). As to the rest of the consumers, they continue paying the same prices for the same products if an increase in $n_a$ happens in Region I. If an increase in $n_a$ happens in Region II or Region III, they enjoy lower prices.

### 3.3 Social Optimum

In all four regions the equilibrium prices fail to induce all consumers on the $n$ spokes to receive their first preferred brands. Take $n_a \in (0, n_a^{III})$ and consider consumers whose first choice is produced by a w-firm (firm $W_i$) and whose second choice is available on Amazon. Not
everybody on $l_i$ will purchase from firm $W_i$. Since $p_w^* - p_a^* > \Delta$, consumers described by

$$x_i \in \left( \frac{1}{2t} (t + \Delta - p_w^* + p_a^*), \frac{1}{2} \right)$$

will purchase from Amazon. This is inefficient, as it results in higher transport costs and lost $\Delta$.

Next, take $n_a \in (n_a^{III}, n)$ and consider consumers whose desired brands (varieties $i$ and $j$) are both sold on Amazon. Because the equilibrium is in mixed strategies, firms $A_i$ and $A_j$ will end up charging different prices. Suppose firm $A_i$’s price is higher, then some consumers on $l_i$ will purchase variety $j$, which is inefficient.

For the $n$ brands to be allocated to consumers efficiently, the following two conditions must be satisfied: (i) each type of firms charges the same price, and (ii) the price difference does not exceed $\Delta$. Any set of efficient prices allows the social planner to achieve the welfare

$$W(n_a) = n_a \frac{1}{2} \left( v - \frac{t}{4} - c \right) + (n - n_a) \frac{1}{2} \left( v + \Delta - \frac{t}{4} - c \right)$$

$$+ (N - n) \frac{1}{2} n_a \frac{n_a}{N - 1} \left( v - \frac{3t}{4} - c \right).$$

(8)

The first term are consumers on the $n_a$ spokes whose first preferred brand is offered on Amazon; they travel $t/4$ on average. The second term are consumers on the $n - n_a$ spokes whose first preferred brand is a product produced by a w-firm; they travel $t/4$ on average. The third term are consumers on the $N - n$ spokes whose first choice is not produced and whose second choice is available on Amazon; they travel $3t/4$ on average.

Since $W(n_a)$ is linear in $n_a$, it is maximized at one of the two extremes. When $n_a$ equals 0 or $n$, condition (ii) is mute, and the equilibrium prices obtained in Lemmas 1 and 2 are efficient. This means that once the social planner sets $n_a$ to 0 or $n$, he/she does not need to adjust the prices.

**Proposition 2 (Social optimum).** The socially optimal number of $a$-firms is

$$n_a^{SO} = \begin{cases} 0, & \text{if } \Delta > \frac{N-n}{N-1} \left( v - \frac{3t}{4} - c \right), \\ n, & \text{if } \Delta < \frac{N-n}{N-1} \left( v - \frac{3t}{4} - c \right). \end{cases}$$

At these two possible values of $n_a^{SO}$ the equilibrium prices induce efficiency.

This is intuitive. A branded website increases the perceived value of the product; the drawback is that consumers who lack their first choices and whose second choice is a variety produced by a w-firm end up not purchasing. When $n = N$, it is socially optimal that all firms sell through their own websites, no matter what $\Delta$ is. When $n < N$, then all firms should sell
through Amazon if ∆ is low, and operate their own websites if ∆ is high.

4 Long Run: Amazon or Website?

In the previous section the composition of firms was fixed. In this section I allow firms to choose between selling through Amazon and operating their own websites. The number of a-firms becomes endogenous.

Suppose all firms operate their own websites. For \( n_a = 0 \) to be an equilibrium, no firm should benefit from becoming an a-firm. We plug \( n_a = 1 \) into

\[
\Pi_a^*(n_a) = \frac{2N - n_a - 1}{2(N - 1)} \left( v - \frac{3t}{2} - c \right)
\]

from Corollary 1 to obtain the profit of the deviating firm:

\[
v - \frac{3t}{2} - c.
\]

We do not want this number to exceed \( \Pi_w^* \) calculated in Lemma 1. That is,

\[
\Pi_w^* = \frac{1}{2} \left( v + \Delta - \frac{t}{2} - c \right) \geq v - \frac{3t}{2} - c,
\]

or

\[
\Delta \geq v - c - \frac{5t}{2}.
\]

The social optimum is achieved, because in the assumed parameter space (5) the above condition is stronger than the condition

\[
\Delta > \frac{N - n}{N - 1} \left( v - \frac{3t}{4} - c \right)
\]

in Proposition 2.

Now suppose ∆ is below \( v - c - 5t/2 \). How many firms will be selling through Amazon? Let \( n_a^* \) be the equilibrium number of a-firms. It must satisfy the following two conditions:

\[
\Pi_a^*(n_a^*) \geq \Pi_w^*(n_a^* - 1) \quad \text{and} \quad \Pi_w^*(n_a^*) \geq \Pi_a^*(n_a^* + 1).
\]

The first condition makes sure an a-firm does not have incentives to become a w-firm, while the second condition says a switch in the other direction is not profitable. For the parameter values used in Figures 2 and 3, we find that \( n_a^* = 21 \).

If we treat \( n_a \) as a continuous variable, the equilibrium number of a-firms will be determined
by

\[ \Pi_w^*(n_a^*) = \Pi_a^*(n_a^*). \]

In Figure 3 the intersection occurs in Region III at \( n_a^* = 22.69 \). The intersection could have happened in Region II, but not in Region I. This is because in Region I \( \Pi_w^*(n_a) \) and \( \Pi_a^*(n_a) \) are linear, and the former is always steeper than the latter:

\[ \Pi_w^*(n_a) = \frac{v + \Delta - \frac{5t}{2} - c}{2(N - 1)} < \Pi_a^*(n_a) = \frac{v - \frac{3t}{2} - c}{2(N - 1)}. \]

When \( \Delta \) is sufficiently close to zero, all firms will be selling through Amazon. For \( n_a = n \) to be an equilibrium, \( \Pi_w^*(n - 1) \) should not exceed \( \Pi_a^*(n) \) from Lemma 2. Because the firms’ equilibrium profits in Region IV are beyond our reach, so is \( \Pi_w(n - 1) \).

Table 1 calculates the equilibrium number of a-firms for various values of \( \Delta \) and \( n \), while the other parameters stay at \( c = 2 \), \( v = 7 \), \( t = 1 \), and \( N = 100 \). The aforementioned \( n_a^* = 22.69 \) (framed) is in the \( \Delta = 1 \), \( n = 90 \) cell. The first row is all zeros because \( n_a^* = 0 \) when

\[ \Delta \geq v - c - \frac{5t}{2} = 7 - 2 - \frac{5 \times 1}{2} = 2.5. \]

An entry of “-” indicates that either \( \Pi_w^*(n_a) \) and \( \Pi_a^*(n_a) \) intersect somewhere in Region IV (the equilibrium of the pricing game is in mixed strategies), or \( \Delta \) is so small that \( n_a^* = n \). All Region II intersections are shaded gray. Because in Region II neither \( \Pi_w^*(n_a) \) nor \( \Pi_a^*(n_a) \) depend on \( n \), the shaded numbers in each row stay the same. The rest are Region III intersections.

The broken diagonal line divides the table into two parameter regions:

\[ \Delta > \frac{N - n}{N - 1} \left( v - \frac{3t}{4} - c \right) \quad \text{and} \quad \Delta < \frac{N - n}{N - 1} \left( v - \frac{3t}{4} - c \right). \]

The threshold

\[ \frac{N - n}{N - 1} \left( v - \frac{3t}{4} - c \right) \]

equals 1.29 when \( n = 70 \), 1.07 when \( n = 75 \), 0.86 when \( n = 80 \), 0.64 when \( n = 85 \), 0.43 when \( n = 90 \), 0.21 when \( n = 95 \), and 0 when \( n = 100 \). The socially optimal number of a-firms is zero above the diagonal; \( n_a^{SO} = n \) below the diagonal (Proposition 2).

Table 1 reveals:

- When \( \Delta \geq v - c - 5t/2 = 2.5 \), the equilibrium number of a-firms coincide with the socially optimal number of a-firms (both equal zero). In all other situations \( n_a^* \) tends to be excessive for higher values of \( n \), and deficient for smaller values of \( n \).

- As \( n \) increases, the equilibrium number of a-firms decreases or, if the equilibrium switches
Table 1: Equilibrium number of a-firms: $n^*_a$

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>70</th>
<th>75</th>
<th>80</th>
<th>85</th>
<th>90</th>
<th>95</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2.3</td>
<td>13.93</td>
<td>13.93</td>
<td>13.93</td>
<td>13.93</td>
<td>13.93</td>
<td>13.93</td>
<td>13.93</td>
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<tr>
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<td>-</td>
<td>17.63</td>
<td>17.45</td>
<td>17.32</td>
<td>17.29</td>
<td>17.29</td>
<td>17.29</td>
</tr>
<tr>
<td>1.5</td>
<td>-</td>
<td>18.41</td>
<td>18.19</td>
<td>18.03</td>
<td>17.91</td>
<td>17.91</td>
<td>17.91</td>
</tr>
<tr>
<td>1.4</td>
<td>-</td>
<td>-</td>
<td>9.01</td>
<td>8.81</td>
<td>8.67</td>
<td>18.57</td>
<td>18.57</td>
</tr>
<tr>
<td>1.3</td>
<td>-</td>
<td>-</td>
<td>9.93</td>
<td>9.68</td>
<td>9.50</td>
<td>19.37</td>
<td>19.28</td>
</tr>
<tr>
<td>1.2</td>
<td>-</td>
<td>-</td>
<td>20.97</td>
<td>20.66</td>
<td>20.44</td>
<td>20.27</td>
<td>20.13</td>
</tr>
<tr>
<td>1.0</td>
<td>-</td>
<td>-</td>
<td>23.53</td>
<td>23.05</td>
<td>22.69</td>
<td>22.42</td>
<td>22.20</td>
</tr>
<tr>
<td>0.9</td>
<td>-</td>
<td>-</td>
<td>25.15</td>
<td>24.54</td>
<td>24.08</td>
<td>23.73</td>
<td>23.45</td>
</tr>
<tr>
<td>0.8</td>
<td>-</td>
<td>-</td>
<td>27.10</td>
<td>26.31</td>
<td>25.71</td>
<td>25.26</td>
<td>24.90</td>
</tr>
<tr>
<td>0.7</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>28.45</td>
<td>27.66</td>
<td>27.06</td>
<td>26.59</td>
</tr>
<tr>
<td>0.6</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>31.13</td>
<td>30.06</td>
<td>29.25</td>
<td>28.63</td>
</tr>
</tbody>
</table>

from Region III to Region II, remains the same. Put differently, there may be fewer firms on Amazon in more concentrated markets. This is counterintuitive; it would be reasonable to expect that as new firms enter the market, some become a-firms and the others w-firms, so both $n^*_a$ and $n - n^*_a$ increase. Here is what actually happens in Region III. Suppose a new firm enters the market as a w-firm. This firm is a direct competitor to the a-firms, but not to the w-firms. As a result, $\Pi^*_a$ goes down more than $\Pi^*_w$ does. This has a snowball effect: Each subsequent entrant will prefer to join the w-firms. As the difference between $\Pi^*_w$ and $\Pi^*_a$ grows, some a-firms would become w-firms to restore the equality of profits in equilibrium. The equilibrium number of a-firms decreases.

- The higher is $\Delta$, the lower is the equilibrium number of a-firms, which is in line with the economics intuition.
- Price dispersion among firms of the same type is more likely for smaller values of $n$ and/or $\Delta$. This interesting result is a consequence of the aforementioned negative effects of $n$ and $\Delta$ on the equilibrium number of a-firms. A decrease in $n$ or $\Delta$ pushes $n^*_a$ into Region IV, where the equilibrium is in mixed strategies.

What is the effect of market concentration on the equilibrium prices? In the normal oligopoly
Table 2: Equilibrium prices: $p^*_w, p^*_a$

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>70</th>
<th>75</th>
<th>80</th>
<th>85</th>
<th>90</th>
<th>95</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.6</td>
<td>-</td>
<td>6.77, 4.33</td>
<td>6.75, 4.23</td>
<td>6.73, 4.15</td>
<td>6.72, 4.12</td>
<td>6.72, 4.12</td>
<td>6.72, 4.12</td>
</tr>
<tr>
<td>1.4</td>
<td>-</td>
<td>-</td>
<td>6.38, 4.15</td>
<td>6.36, 4.06</td>
<td>6.35, 3.99</td>
<td>6.33, 3.92</td>
<td>6.33, 3.92</td>
</tr>
<tr>
<td>0.8</td>
<td>-</td>
<td>-</td>
<td>5.15, 3.84</td>
<td>5.17, 3.77</td>
<td>5.18, 3.71</td>
<td>5.19, 3.65</td>
<td>5.20, 3.60</td>
</tr>
<tr>
<td>0.6</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>4.72, 3.65</td>
<td>4.74, 3.59</td>
<td>4.76, 3.54</td>
<td>4.77, 3.49</td>
</tr>
</tbody>
</table>

In the case of Chen and Riordan (2007) the higher is $n$, the lower is the equilibrium price. Will the equilibrium prices be decreasing functions of $n$ in the present model, with $n_a$ determined endogenously?

I calculated the equilibrium prices for the rows $\Delta = 0.6$, $\Delta = 0.8$, $\Delta = 1.4$, and $\Delta = 1.6$ in Table 1, and recorded them in Table 2. The prices in the $\Delta = 1.6$, $n = 90$ cell coincide with those in the $\Delta = 1.6$, $n = 95$ and $\Delta = 1.6$, $n = 100$ cells. This is because the three cells share the same $n^*_a = 17.29$, and the prices in Region II are independent of the total number of firms (Proposition 1). The same can be said about the two shaded cells in the $\Delta = 1.4$ row.

In the $\Delta = 1.4$ and $\Delta = 1.6$ rows both $p^*_w$ and $p^*_a$ go down and then plateau as $n$ increases. In the $\Delta = 0.6$ and $\Delta = 0.8$ rows, $p^*_w$ goes up while $p^*_a$ goes down as $n$ increases. The unusual result that a price can be increasing in the total number of firms is a consequence of the negative relationship between $n^*_a$ and $n$.

## 5 Concluding Remarks

In this theoretical paper I considered a setting in which firms can sell their products through an online marketplace (Amazon Marketplace) or through their own websites. While selling through Amazon expands a firm’s customer reach, running a website can help the firm to build brand equity. I derived the equilibrium prices and profits, analyzed the firms’ behavior in the long run, and compared the equilibrium outcome with the social optimum.

I used the spokes model of spatial competition proposed by Chen and Riordan (2007) because it can easily accommodate more than one type of firms. The reason being that competition in the spokes model is non-localized. The Salop’s circular city model would not work, because with more than one type of firms there are numerous distinct ways of locating the firms equidistantly around the circle. Chen and Riordan’s assumption that each consumer cares about a limited number of brands is important to the present study, too. When some w-firms become
a-firms, consumers whose first choice is not produced and whose second choice was produced by a w-firm and now is produced by an a-firm now obtain a positive surplus. This expansion effect allows for a detailed study of how the composition of firms affects prices, consumers, firms and welfare.

My approach to modeling w-firms as firms that non-fans do not see, as opposed to a-firms, is novel and offers new insights. Despite the fact that I restricted my attention to the parameter space that coincides with the normal oligopoly case of Chen and Riordan (2007) when $n_a = n$ with “foreseeable” comparative statics, I obtained a few surprising results. For example, I found that the price that w-firms charge can be increasing in the total number of firms and that a symmetric pure-strategy Nash equilibrium might not exist. The latter means price dispersion among the firms of the same type.

I abstracted from costs associated with creating and running a website, as well as the fees that Amazon charges its third-party retailers.\(^5\) Incorporating these costs will make the model more complex, taking the focus away from the strategic effects of Amazon Marketplace. I also assumed that a firm cannot be an a-firm and a w-firm at the same time. In reality, some companies sell through both rented and owned platforms. These are firms that are capable of synchronizing the inventory over multiple channels; many of them started out on a rented platform like Amazon to make some sales until their own branded websites kick into high gear. Once you created a website and invested in building your brand, you might not want it to be commoditized with the rest of the listings on Amazon.

References


\(^5\)Amazon offers individual seller and pro merchant accounts. Individual seller accounts are geared to occasional, low-volume sellers. Pro merchant accounts are designed to meet the needs of businesses and provide many volume-selling features for $39.99 per month fee, including unlimited product listings and inventory management. All Amazon sellers, both individual and pro, pay a referral fee for every item that sells on Amazon. Referral fees range from 6% up to 20% depending on the product’s category.


Appendix

Proof of Lemma 2

Let \( p_{a}^{*} \) be the price that a-firms charge in equilibrium. Firm \( A_{i} \) faces the demand

\[
\frac{n-1}{N-1} \left( \frac{1}{2t} (t - p_{i} + p_{a}^{*}) + \frac{N-n}{N-1} \right)
\]
in the neighborhood of \( p_{a}^{*} \). The profit of firm \( A_{i} \) is, therefore,

\[
\left( \frac{n-1}{N-1} \left( \frac{1}{2t} (t - p_{i} + p_{a}^{*}) + \frac{N-n}{N-1} \right) \right) (p_{i} - c),
\]
or

\[
\frac{1}{2(N-1)} \left( \frac{n-1}{t} (t - p_{i} + p_{a}^{*}) + 2(N-n) \right) (p_{i} - c).
\]

(9)

The first-order condition is

\[
\frac{n-1}{t} (t - p_{i} + p_{a}^{*}) + 2(N-n) - \frac{n-1}{t} (p_{i} - c) = 0.
\]

(10)

Substituting \( p_{i} = p_{a}^{*} \) and yields

\[
2N - n - 1 - \frac{n-1}{t} (p_{a}^{*} - c) = 0,
\]

\[
p_{a}^{*} = c + \frac{2N - n - 1}{n-1} t.
\]

We can use (9) and (10) to calculate the equilibrium profit:

\[
\Pi_{a}^{*} = \frac{n-1}{2(N-1)t} (p_{a}^{*} - c)^{2} = \frac{(2N - n - 1)^{2}}{2(N-1)(n-1)} t.
\]

Proof of Lemma 3

That \( p_{w}^{*} - p_{a}^{*} < \Delta \) cannot happen in equilibrium is obvious. If a w-firm increases its price to \( p_{a}^{*} + \Delta \), the number of consumers it attracts will not change. Thus, the deviation is profitable.

Now suppose \( p_{w}^{*} - p_{a}^{*} = \Delta \). Under such prices each firm serves consumers located on the same spoke; a-firms also serve consumers who lack their first choices but whose second choices are
available on Amazon. Consider a deviation by an a-firm to $p_a^* - \epsilon$. The firm will lose

$$\epsilon \left( \frac{1}{2} + \frac{1}{2} \frac{N-n}{N-1} \right)$$

on its existing customers and gain

$$\frac{\epsilon}{2t} \frac{n-1}{N-1} (p_a^* - \epsilon - c)$$

on the customers it steals from the other firms. Equilibrium requires

$$\epsilon \left( \frac{1}{2} + \frac{1}{2} \frac{N-n}{N-1} \right) \geq \frac{\epsilon}{2t} \frac{n-1}{N-1} (p_a^* - \epsilon - c)$$

for all $\epsilon > 0$, or

$$p_a^* \leq c + \frac{2N-n-1}{n-1} t.$$

Next, consider a deviation by an a-firm to $p_a^* + \epsilon$. The firm will gain

$$\epsilon \left( \frac{1}{2} + \frac{1}{2} \frac{N-n}{N-1} \right)$$

on its existing customers and lose

$$\frac{\epsilon}{2t} \frac{n_a-1}{N-1} (p_a^* + \epsilon - c)$$

as some of its customers switch to the other a-firms. Equilibrium requires

$$\epsilon \left( \frac{1}{2} + \frac{1}{2} \frac{N-n}{N-1} \right) \leq \frac{\epsilon}{2t} \frac{n_a-1}{N-1} (p_a^* + \epsilon - c)$$

for all $\epsilon > 0$, or

$$p_a^* \geq c + \frac{2N-n-1}{n_a-1} t.$$

It is easy to see that the two conditions cannot be satisfied simultaneously.

**Proof of Proposition 1**

We prove each of the three parts in turn.

(i) When $n_a \in [0, n_a^l)$, w-firm adopt the same strategy as in the case of $n_a = 0$,

$$p_w^* = v + \Delta - \frac{t}{2}.$$
A-firms focus on capturing the consumers whose first choice is produced by a w-firm, so they set their prices equal to

\[ p^*_a = p^*_w - t - \Delta = v - \frac{3t}{2}. \]

The resulting profits are

\[ \Pi^*_w = \frac{1}{2} N - \frac{n_a - 1}{N - 1} (p^*_w - c) = \frac{N - n_a - 1}{2(N - 1)} \left( v + \Delta - \frac{t}{2} - c \right) \]

and

\[ \Pi^*_a = \left( \frac{1}{2} + \frac{1}{2} \frac{N - n_a}{N - 1} \right) (p^*_a - c) = \frac{2N - n_a - 1}{2(N - 1)} \left( v - \frac{3t}{2} - c \right). \]

(ii) Like in Region I, in Region II a-firms focus on capturing the consumers whose first choice is produced by a w-firm, so they set their prices equal to \( p^*_a = p^*_w - t - \Delta \). Firm \( W_i \) faces the demand

\[ \frac{n_a}{(N - 1)} \frac{1}{2t} (t + \Delta - p_i + (p^*_w - t - \Delta)) + \frac{1}{2} \frac{N - n_a - 1}{N - 1} \]

in the neighborhood of \( p^*_w \). The profit of firm \( W_i \) is, therefore,

\[ \left( \frac{n_a}{(N - 1)} \frac{1}{2t} (t + \Delta - p_i + (p^*_w - t - \Delta)) + \frac{1}{2} \frac{N - n_a - 1}{N - 1} \right) (p_i - c), \]

or

\[ \frac{1}{2(N - 1)} \left( \frac{n_a}{t} (-p_i + p^*_w) + N - n_a - 1 \right) (p_i - c). \] (11)

The first-order condition is

\[ \frac{n_a}{t} (-p_i + p^*_w) + N - n_a - 1 - \frac{n_a}{t} (p_i - c) = 0. \] (12)

Substituting \( p_i = p^*_w \) yields

\[ N - n_a - 1 - \frac{n_a}{t} (p^*_w - c) = 0, \]

\[ p^*_w = c + \frac{N - n_a - 1}{n_a} t. \]

Thus,

\[ p^*_a = p^*_w - t - \Delta = c + \frac{N - 2n_a - 1}{n_a} t - \Delta. \]
We can use (11) and (12) to calculate \( \Pi_w^* \):

\[
\Pi_w^* = \frac{n_a}{2(N-1)}(p_w^* - c)^2 = \frac{(N - n_a - 1)^2}{2(N-1)n_a}. 
\]

As to a-firms, they earn

\[
\Pi_a^* = \left( \frac{1}{2} + \frac{1}{2} \frac{N - n_a}{N - 1} \right) (p_a^* - c) = \frac{2N - n_a - 1}{2(N-1)} \left( \frac{N - 2n_a - 1}{n_a} - t - \Delta \right). 
\]

The threshold \( n'_a \) can be obtained by setting \( p_a^* \) to \( v + \Delta - t/2 \):

\[
c + \frac{N - n_a - 1}{n_a} t = v + \Delta - \frac{t}{2},
\]

\[
n'_a = \frac{N - 1}{t} + \frac{1}{2}.
\]

(iii) In Region III \( \Delta < p_w^* - p_a^* < t + \Delta \), so \( \hat{x} \in (0, 1/2) \). Firm \( A_i \) faces the demand

\[
\frac{n_a - 1}{N - 1} \frac{1}{2t} (t - p_i + p_a^*) + \frac{n - n_a}{N - 1} \frac{1}{2t} (p_w^* - p_i - \Delta) + \frac{1}{2} \frac{N - n}{N - 1} + \frac{1}{2} \frac{N - n}{N - 1}
\]

in the neighborhood of \( p_a^* \). The profit of firm \( A_i \) is, therefore,

\[
\left( \frac{n_a - 1}{N - 1} \frac{1}{2t} (t - p_i + p_a^*) + \frac{n - n_a}{N - 1} \frac{1}{2t} (p_w^* - p_i - \Delta) + \frac{1}{2} \frac{N - n}{N - 1} + \frac{1}{2} \frac{N - n}{N - 1} \right) (p_i - c),
\]

or

\[
\frac{1}{2(N-1)} \left( \frac{n_a - 1}{t} (-p_i + p_a^*) + \frac{n - n_a}{t} (p_w^* - p_i - \Delta) + 2N - n - 1 \right) (p_i - c). 
\]

The first-order condition is

\[
n_a - \frac{1}{t} (-p_i + p_a^*) + \frac{n - n_a}{t} (p_w^* - p_i - \Delta) + 2N - n - 1 - \frac{n - 1}{t} (p_i - c) = 0. 
\]

Substituting \( p_i = p_a^* \) yields

\[
\frac{n - n_a}{t} (p_w^* - p_a^* - \Delta) + 2N - n - 1 - \frac{n - 1}{t} (p_a^* - c) = 0,
\]

or

\[
\frac{2n - n_a - 1}{n - 1} p_a^* - \frac{n - n_a}{n - 1} \frac{1}{t} p_w^* = c + \frac{2N - n - 1}{n - 1} \frac{1}{t} - \frac{n - n_a}{n - 1} \Delta.
\]
Firm $W_i$ faces the demand

\[ \frac{n_a}{N-1} \left( t + \Delta - p_i + p_a^* \right) + \frac{1}{2} \frac{N - n_a - 1}{N - 1} \]

in the neighborhood of $p_w^*$. The profit of firm $W_i$ is, therefore,

\[ \left( \frac{n_a}{N-1} \left( t + \Delta - p_i + p_a^* \right) + \frac{1}{2} \frac{N - n_a - 1}{N - 1} \right) (p_i - c), \]

or

\[ \frac{1}{2(N-1)} \left( \frac{n_a}{t} (\Delta - p_i + p_a^*) + N - 1 \right) (p_i - c). \] (15)

The first-order condition is

\[ \frac{n_a}{t} (\Delta - p_i + p_a^*) + N - 1 - \frac{n_a}{t} (p_i - c) = 0. \] (16)

Substituting $p_i = p_w^*$ yields

\[ \frac{n_a}{t} (\Delta - p_w^* + p_a^*) + N - 1 - \frac{n_a}{t} (p_w^* - c) = 0, \]

or

\[ 2p^*_w - p_a^* = c + \frac{N - 1}{n_a} t + \Delta. \]

To find $p_w^*$ and $p_a^*$, we solve the system of equations

\[
\begin{aligned}
2n - n_a - 1 &- p_a^* - n - n_a - p_w^* = c + \frac{2N - n - 1}{n - 1} t - \frac{n - n_a}{n - 1} \\
2p_w^* - p_a^* = c + \frac{N - 1}{n_a} t + \Delta
\end{aligned}
\]

Let

\[ \gamma = \frac{n - n_a}{n - 1}, \]

\[ \Phi = \frac{2N - n - 1}{n - 1} t - \frac{n - n_a}{n - 1} \Delta, \]

\[ \Psi = \frac{N - 1}{n_a} t + \Delta. \]

Then the system can be rewritten as

\[
\begin{aligned}
(1 + \gamma)p_a^* - \gamma p_w^* &= c + \Phi \\
2p_w^* - p_a^* &= c + \Psi
\end{aligned}
\]
Solving it yields

\[ p^*_w = c + \frac{\Phi + (1 + \gamma)\Psi}{2 + \gamma} = c + \frac{2N-n-1}{n-1}t - n - 1 \frac{n-n_a}{n_a} \Delta + \left(1 + \frac{n-n_a}{n-1} \right) \left( \frac{N-1}{n_a} t + \Delta \right) 2 + \frac{n-n_a}{n-1} \]
\[ = c + \frac{(N - n + (N-1)(2n-1)) t + \Delta}{3n - n_a - 1} \]

and

\[ p^*_a = c + \frac{2\Phi + \gamma\Psi}{2 + \gamma} = c + \frac{2 \left( \frac{2N-n-1}{n-1} t - n - 1 \frac{n-n_a}{n_a} \Delta \right) + \frac{n-n_a}{n-1} \left( \frac{N-1}{n_a} t + \Delta \right)}{2 + \frac{n-n_a}{n-1}} \]
\[ = c + \frac{3N - 2n - 1 + (N-1)n}{n_a} t - (n - n_a) \Delta}{3n - n_a - 1} . \]

We want to make sure that \( p^*_w - p^*_a < t + \Delta \):

\[ c + \frac{\Phi + (1 + \gamma)\Psi}{2 + \gamma} - c - \frac{2\Phi + \gamma\Psi}{2 + \gamma} < t + \Delta, \]
\[ \Psi - \Phi < (t + \Delta)(2 + \gamma). \]

Substituting for \( \gamma \), \( \Phi \) and \( \Psi \) yields

\[ \frac{N-1}{n_a} t + \Delta - \frac{2N-n-1}{n-1} t + \frac{n-n_a}{n-1} \Delta < (t + \Delta) \left( 2 + \frac{n-n_a}{n-1} \right), \]
\[ \frac{N-1}{n_a} t - \frac{2N-n-1}{n-1} t < \Delta + t \left( 2 + \frac{n-n_a}{n-1} \right), \]
\[ n_a > \frac{N-1}{\Delta + 2N+2n-n_a-3}. \]

The threshold \( n_a^{II} \) can be found from

\[ n_a^{II} = \frac{N-1}{\Delta + 2N+2n-n_a^{II}-3}, \]
\[ n_a^{II} = N + n - 3 \frac{\Delta}{2} + \frac{(n-1)\Delta}{2t} - \sqrt{\left( N + n - 3 \frac{\Delta}{2} + \frac{(n-1)\Delta}{2t} \right)^2 + (N-1)(n-1)}. \]
We can use (15) and (16) to calculate the profit of a \( w \)-firm in equilibrium:

\[
\Pi_w^* = \frac{n_a}{2(N-1)t} (p_w^* - c)^2 = \frac{n_a}{2(N-1)t} \left( \frac{N - n + \frac{(N-1)(2n-1)}{n_a}}{3n - n_a - 1} t + \Delta \right)^2.
\]

Similarly, we can use (13) and (14) to calculate the profit of an \( a \)-firm in equilibrium:

\[
\Pi_a^* = \frac{n-1}{2(N-1)t} (p_a^* - c)^2 = \frac{n-1}{2(N-1)t} \left( \frac{3N - 2n - 1 + \frac{(N-1)n}{n_a}}{3n - n_a - 1} \right)^2.
\]

Finally, we want to make sure that firm \( A_i \) does not have incentives to deviate to a significantly higher price \( p_i \in (p_w^* - \Delta, p_a^* + t) \), in which case it would be maximizing

\[
\left( \frac{n_a - 1}{N-1} \frac{1}{t} (t - p_i + p_a^*) + \frac{1}{2} \frac{N - n}{N - 1} + \frac{1}{2} \frac{N - n}{N - 1} \right) (p_i - c),
\]

or

\[
\frac{1}{2(N-1)} \left( \frac{n_a - 1}{t} (-p_i + p_a^*) + 2N - n - 1 \right) (p_i - c).
\]  

(17)

The first-order condition is

\[
\frac{n_a - 1}{t} (-p_i + p_a^*) + 2N - n - 1 - \frac{n_a - 1}{t} (p_i - c) = 0,
\]

implying

\[
p_i = \frac{p_a^* - c + 2N - n - 1}{2(n_a - 1)} t.
\]

We use (17) and (18) to calculate the profit of the deviating firm:

\[
\Pi'_a = \frac{n_a - 1}{2(N-1)t} (p_i - c)^2 = \frac{n_a - 1}{2(N-1)t} \left( \frac{p_a^* - c}{2} + \frac{2N - n - 1}{2(n_a - 1)} t \right)^2.
\]

The threshold \( n_a^{III} \) is defined by equating \( \Pi'_a \) and \( \Pi_a^* \),

\[
\frac{n_a - 1}{2(N-1)t} \left( \frac{p_a^* - c}{2} + \frac{2N - n - 1}{2(n_a - 1)} t \right)^2 = \frac{n - 1}{2(N-1)t} \left( p_a^* - c \right)^2,
\]

or

\[
(n_a - 1) \left( \frac{p_a^* - c}{2} + \frac{2N - n - 1}{2(n_a - 1)} t \right)^2 = (n - 1) (p_a^* - c)^2,
\]

or
where

$$p_a^* = c + \frac{(3N - 2n - 1 + \frac{(N-1)n}{n_a}) t - (n - n_a)\Delta}{3n - n_a - 1}.$$