Verifying Curvature of Profit and Cost/Expenditure Functions

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Abstract. Convexity or concavity of optimal value functions is sometimes checked by evaluating leading principal minors of the Hessian matrix. This practice is not justified by extant theorems about semidefinite matrices. A theorem is presented that justifies the practice and provides a relatively easy method of proving the relationship between semidefiniteness and principal minors.

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1 Introduction

Suppose the function $\phi\colon P \to \mathbb{R}^1$ ($P \subset \mathbb{R}^n_+$) has arisen in an applied theory or empirical investigation and the researcher wants to check whether ϕ is an optimal value function for a price-taking economic actor. Duality theory establishes that ϕ is indeed a profit (cost/expenditure) function provided only that P is an open convex cone on which ϕ is linear homogeneous and convex (concave).[1] Linear homogeneity is simple to check, but convexity (concavity) can be more challenging even if ϕ is C^2. Extant theorems establish equivalence between convexity (concavity) and global positive (negative) semidefiniteness of the Hessian, and between positive (negative) semidefinteness of a Hermitian matrix and nonnegativity (signs $(-1)^k$ or zero, where k is the order) of all principal minors. Use of these theorems therefore entails the formidable task of checking signs for **all** $2^n - 1$ principal minors. Varian [13, pp. 12 – 14] checks signs of only the n leading principal minors, which would be justified by the equivalence between positive (negative) definiteness of a Hermitian matrix and positivity (signs $(-1)^k$) of the leading principal minors, except that linear homogeneity implies the Hessian is singular and therefore *cannot* be definite.[2] This short paper provides a theorem that justifies Varian's approach. A byproduct is a relatively easy proof of the standard result that weak signs of all principal minors is sufficient for semidefiniteness of a Hermitian matrix.

2 Main Theorem

First a bit of notation:

- $x \overset{s}{=} y$ means the real numbers x and y have the same sign.
- Given a matrix A with complex entries a_{ij}, A^* denotes the conjugate transpose of A (the (j,i) entry of A^* is the complex conjugate of a_{ij}). Recall that, if $A \in \mathbb{C}^n$ (a vector), then A^*A is real and nonnegative, and is zero if and only if $A = 0$.
- Given an $n \times n$ complex matrix A, let $J \subset \{1, \ldots, n\}$ denote an index set of the rows and columns of A ($J \neq \emptyset$). A_J denotes the principal submatrix of A consisting of the rows and columns in J, and $\#J$ denotes the number of elements in J (the order of the principal submatrix). In the special cases $J = \{1, \ldots, i\}$ for $i = 1, \ldots, n$, A_J is the i^{th} order leading principal submatrix of A. The determinant $|A_J|$ is a principal minor of order $\#J$. Enumerating the elements of J in ascending order as $J = \{j_1, \ldots, j_{\#J}\}$ yields a permutation matrix

[1] In the cost/expenditure case ϕ must also be nonnegative and typically $P = \mathbb{R}^n_{++}$. Some duality treatments of cost/expenditure assume ϕ is nondecreasing (e.g., Varian [12, pp. 84 – 85], Jehle and Reny [6, Theorem 2.2], Kreps [7, Proposition 10.15]), but this is a redundant property because every linear homogeneous and concave nonnegative function on $\mathbb{R}^n_{++}$ is nondecreasing. Continuity and differentiability are sometimes mentioned as well. The former is redundant when P is open because every convex (concave) function on an open domain is continuous. The latter is only for convenience. There are additional requirements on the behavior of ϕ as a function of utility for the expenditure case, which are omitted because the discussion herein concerns ϕ exclusively as a function of prices.

[2] Varian always finds the determinant of the Hessian is zero.

P_J with entries $p_{ij} = 0$ for $i, j = 1, \ldots, n$; except $p_{ij_i} = 1$ for $i = 1, \ldots, \#J$ and $p_{ii} = 1$ for $i = \#J + 1, \ldots, n$. When symmetrically applied to A, P_J makes A_J a leading principal submatrix. As with all permutation matrices, $P_J^* P_J = I_n$.

Theorem 1. *If the $n \times n$ Hermitian matrix A has a principal submatrix A_J satisfying:*

1. *A_J is positive (negative) definite,*
2. *No higher order principal submatrix of A is positive (negative) definite, and*
3. *$|A_K| \geq 0$ ($(-1)^{\#K}|A_K| \geq 0$) for every K satisfying $J \subset K \subset \{1, \ldots, n\}$;*

then A is positive (negative) semidefinite.

Proof. Begin with the positive semidefinite case. If $A_J = A$ then A is trivially positive semidefinite, so assume A_J excludes some rows and (the same) columns. Then:

$$P_J A P_J^* = \begin{bmatrix} A_J & B \\ B^* & C \end{bmatrix}$$

for some matrices B and C (C is Hermitian). Consider an arbitrary column b_i from B and the corresponding diagonal entry c_{ii} from C. $|A_J| > 0$ due to assumption 1. Using the standard formula for the determinant of a partitioned matrix:

$$\begin{vmatrix} A_J & b_i \\ b_i^* & c_{ii} \end{vmatrix} = |A_J||c_{ii} - b_i^* A_J^{-1} b_i| \stackrel{s}{=} c_{ii} - b_i^* A_J^{-1} b_i.$$

This is nonnegative due to assumption 3. If it is positive then the principal submatrix under consideration is positive definite, contradicting assumption 2, so $c_{ii} - b_i^* A_J^{-1} b_i = 0$. Now consider any two columns from B and the corresponding 2×2 submatrix from C:

$$\begin{aligned} \begin{vmatrix} A_J & b_i & b_j \\ b_i^* & c_{ii} & c_{ij} \\ b_j^* & c_{ij}^* & c_{jj} \end{vmatrix} &= |A_J| \left| \begin{bmatrix} c_{ii} & c_{ij} \\ c_{ij}^* & c_{jj} \end{bmatrix} - \begin{bmatrix} b_i^* \\ b_j^* \end{bmatrix} A_J^{-1} \begin{bmatrix} b_i & b_j \end{bmatrix} \right| \\ &\stackrel{s}{=} \begin{vmatrix} 0 & c_{ij} - b_i^* A_J^{-1} b_j \\ c_{ij}^* - b_j^* A_J^{-1} b_i & 0 \end{vmatrix} \\ &= -(c_{ij} - b_i^* A_J^{-1} b_j)(c_{ij}^* - b_j^* A_J^{-1} b_i) \\ &= -(c_{ij} - b_i^* A_J^{-1} b_j)(c_{ij} - b_i^* A_J^{-1} b_j)^*. \end{aligned}$$

This is again nonnegative due to assumption 3, which implies $c_{ij} - b_i^* A_J^{-1} b_j = 0$ for every i, j (including $i = j$). Therefore $C = B^* A_J^{-1} B$. Recalling that every positive (semi) definite (Hermitian) matrix has a unique positive (semi) definite (Hermitian) square root matrix, we may define $F = \begin{bmatrix} A_J^{1/2} & A_J^{-1/2} B \end{bmatrix}$ and obtain:

$$F^* F = \begin{bmatrix} A_J^{1/2} \\ B^* A_J^{-1/2} \end{bmatrix} \begin{bmatrix} A_J^{1/2} & A_J^{-1/2} B \end{bmatrix} = \begin{bmatrix} A_J & B \\ B^* & B^* A_J^{-1} B \end{bmatrix} = \begin{bmatrix} A_J & B \\ B^* & C \end{bmatrix}.$$

For any nonzero $x \in \mathbb{C}^n$, let $y(x) = FP_J x$. Then:

$$x^* A x = x^* P_J^* F^* F P_J x = y(x)^* y(x) \geq 0.$$

Hence A is positive semidefinite.

For the negative semidefinite case, replace A_J in the above argument by $-A_J$ and A by $-A$, noting that (1) $-A_J$ is positive definite, (2) no higher order principal submatrix of $-A$ is positive definite, and (3) $|-A_K| = (-1)^{\#K}|A_K| \geq 0$ for every K satisfying $J \subset K \subset \{1, \ldots, n\}$. The conclusion is $-A$ is positive semidefinite, or A is negative semidefinite. □

3 Applications

Although the Hessian is singular in many duality applications, it is often possible to establish that an $(n-1)^{\text{st}}$ order principal submatrix is definite by checking only its $(n-1)$ leading principal minors, a much easier task than checking all $2^n - 1$ principal minors of the Hessian. Theorem 1 shows that doing so suffices to establish semidefiniteness of the Hessian.

Corollary 1. *Assume A is an $n \times n$ singular Hermitian matrix. If A has an $(n-1)^{st}$ order positive (negative) definite principal submatrix A_J then A is positive (negative) semidefinite.*

Proof. The only principal submatrix of higher order than A_J is A, and $|A| = 0$. Apply Theorem 1. □

Corollary 1 is the first explicit proof that Varian's [13, pp. 12 – 14] approach to checking convexity (concavity) is correct. Note also that Theorem 1 can be used when the Hessian does not have a definite $(n-1)^{\text{st}}$ order principal submatrix. For example, if it is discovered by checking leading principal minors that $(n-2)$ is the order of the largest definite principal submatrix, then it suffices to verify that all $(n-1)^{\text{st}}$ order principal minors have the correct (weak) sign (because the sole n^{th} order principal minor is zero).

A second application of Theorem 1 easily proves the standard sufficient conditions for semidefiniteness of a Hermitian matrix. Although these conditions are well-known, proofs in the economics literature are scarce. Standard mathematical economics books including Chiang [2, pp. 320 – 323], de la Fuente [4, p. 270], Lancaster [8, pp. 297 – 300], Simon and Blume [9, Theorem 16.2], Sydsæter et al. [10, Theorem 1.7.1] and Takayama [11, Theorem 1.E.11] do not prove the semidefinite case. Extant proofs typically work directly with the quadratic form or characteristic equation and use a limit argument (e.g., Debreu [3], Abadir and Magnus [1, p. 223], Gantmacher [5, p. 307]), making them relatively inaccessible to many economists. The result is effortlessly obtained from Theorem 1.

Corollary 2. *If every principal minor of a Hermitian matrix A is nonnegative (of order $\#J$ has sign $(-1)^{\#J}$ or is zero) then A is positive (negative) semidefinite.*

Proof. If every diagonal entry of A is zero then:

$$0 \leq \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ij}^* & a_{jj} \end{vmatrix} = -a_{ij}a_{ij}^* \text{ for every } i, j = 1, \ldots, n.$$

This implies $a_{ij} = 0$ for every i, j. That is, A is a matrix of zeros, and is therefore trivially positive (negative) semidefinite. So assume there is a nonzero diagonal entry. That entry is a positive (negative) definite principal submatrix, so A has a highest order positive (negative) definite principal submatrix A_J satisfying the assumptions of Theorem 1. □

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